Bažant, Z.P., Adley, M.D., and Xiang, Y. (1996). "Finite strain analysis of deformations of quasibrittle material during missile impact and penetration." in *Advances in Failure Mechanisms in Brittle Materials* (ASME Int. Mech. Engrg. Congress, Atlanta), MD–Vol 75/AMD–Vol. 219, ed. by R.J. Clifton and H.D. Espinosa, publ. by Am. Soc. of Mech. Engrs., New York, pp. 163–169.

MD-Vol. 75
AMD-Vol. 219

ADVANCES IN FAILURE MECHANISMS IN BRITTLE MATERIALS

presented at

THE 1996 ASME INTERNATIONAL
MECHANICAL ENGINEERING CONGRESS AND EXPOSITION
NOVEMBER 17– 22, 1996
ATLANTA, GEORGIA

sponsored by

THE MATERIALS DIVISION, ASME
THE APPLIED MECHANICS DIVISION, ASME

edited by

R. J. CLIFTON
BROWN UNIVERSITY

H. D. ESPINOSA
PURDUE UNIVERSITY

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
United Engineering Center / 345 East 47th Street / New York, N.Y. 10017

MD-Vol. 75/AMD-Vol. 219, Advances in Failure Mechanisms in Brittle Materials
ASME 1996

FINITE STRAIN ANALYSIS OF DEFORMATIONS OF QUASIBRITTLE MATERIAL DURING MISSILE IMPACT AND PENETRATION

Zdeněk P. Bažant,* Fellow. ASME, Mark D. Adley,† and Yuyin Xiang*

* Department of Civil Engineering
Northwestern University. Evanston, Illinois 60208

† U.S. Army Engineers Waterways Experiment Station
Vicksburg, Mississippi 31980-6199

ABSTRACT

The conference presentation deals with three problems involved in finite element analysis of the impact of missiles into reinforced concrete walls and their penetration through the walls: (1) Formulation of the constitutive law for complex nonlinear triaxial behavior of concrete, including the strain-softening damage; (2) extension of the formulation to very large finite strains; and (3) application of the model in dynamic finite element analysis. Only problem (2) is discussed in some detail in this brief paper. Because the Biot strain tensor has a clear physical meaning even for very large finite strains, its use is preferable in the fitting of complex triaxial test data. It is shown that the constitutive relation can be conveniently formulated as a relation of the Biot strain tensor to the back-rotated Cauchy stress tensor, and the justification of this form of the constitutive relation is given.

INTRODUCTION

Impact of missiles and their penetration through concrete walls generates strains of the order of 100% near the missile. The constitutive law used in the analysis must be applicable to such enormous strains. For some metal-forming problems, adequate incremental formulations in an updated Lagrangean frame of reference have been obtained by finite strain generalization of constitutive models of incremental plasticity, such as von Mises plasticity. These formulations are well established.

Such formulations have been tried for concrete, but with little success. The main reason is that the constitutive law of concrete is much more complex. A sophisticated nonlinear triaxial constitutive model which that been shown to give excellent results for concrete at small strains is the microplane model. Its available form, however, requires a total rather than incremental Lagrangean frame of reference. Hibbitt et al . (1994) use in ABAQUS a hyperleastic constitutive law in the form of a relation of the Cauchy stress tensor to the left Cauchy-Green strain tensor. But such a formulation does not seem possible for concrete. As far as the total Lagrangean formulations are concerned, the available models deal mainly with elastomers, which do not exhibit damage and strain-softening and can be formulated on the basis of a simple, easily identified, elastic potential, such as a low-order polynomial in the principal stretches. This is not possible for concrete. An updated Lagragean formulation for hyperelastic treatment of plastic metals has been introduced by Gabriel and Bathe (1995) and Eterovich and Bathe (1990).

An engineer who is trying to fit the test data for concrete and calibrate the constitutive model needs to have a good intuitive understanding of the physical meaning of the strain and stress components used. The simplest finite strain tensor and objective strain tensor are the Green-Lagrange (GL) strain tensor ϵ and the second Piola-Kirchhoff (2PK) stress tensor σ. An extension of a complex constitutive law to finite strain based on these tensors has recently been achieved for concrete in the standard form of a relation of 2PK to GL (Bažant et al. 1996a,b, Bažant 1996). However, this approach to the formulation of the constitutive model works only for moderately large strains of the order of 10%. It does not work for very large strains of the order of 100%. The purpose of this paper is to briefly outline a formulation in progress that appears workable even for very large strains.

Because of the quadratic terms in the expression for the GL tensor, the physical meanings of GL and 2PK at very large strains are not easy to understand. It appears very difficult to employ these tensors in any data fitting endeavor in which the material parameters need to be intuitively adjusted. We will show here a formulation utilizing the Biot strain tensor (the deviation of the right stretch tensor from the unit tensor) whose physical meaning is easy to understand because, in absence of material rotations, its expressions are identical to the engineering strain (linearized strain). The conference presentation will also include a discussion of the constitutive model, which consists of a generalization of the microplane model and a recent finite strain formulation with a additive split of the volumetric and deviatoric strain components.

Finally, the conference presentation will also review extensive large strain finite element computations for missile impact with the microplane model which appeared to be computationally effective. These computations have used a constitutive law that relates the back-rotated Cauchy stress tensor to the Green-Lagrange strain tensor. The recent ones have also used a law that relates this stress tensor to the Biot strain tensor.

FINITE STRAIN AND GENERALIZATION OF COMPLEX CONSTITUTIVE LAW

The finite-strain constitutive law for materials such as concrete may be obtained from the small-strain constitutive law by replacing the small (linearized) strain tensor with one of the infinitely many possible finite strain tensors and then introducing further material parameters which influence only the large strain response. Mathematically the simplest formulation is obtained by choosing the GL tensor. Normally, the stress to be used in the constitutive relation is the work-conjugate stress tensor, which is in this case the 2PK tensor. However, as already mentioned, this causes difficulties in data fitting. But the use of the work-conjugate stress tensor is necessary only for calculating the work. For other purposes, the stress tensor to be defined by the constitutive relation need not be the work-conjugate stress tensor, although its use must not violate the material symmetries.

A practically reasonable form of the constitutive relation can be figured out if the strain tensor is either the linearized (small) strain tensor e, also called the engineering strain, or Hencky's logarithmic strain tensor, and if the stress tensor is the Cauchy (true) stress tensor, although the latter may only be used in a back-rotated form so as to satisfy the objectivity requirements. However, the logarithmic strain is expensive to use, and its use is better avoided. The reason is that, if the material rotation is nonzero, the spectral decomposition must be used to calculate the logarithmic strain tensor and to convert its conjugate stress to 2PK (Bažant 1996). In a large three-dimensional finite element program with many loading steps, the need to calculate the principal strains and their directions carries a significant penalty in terms of computer time because this calculation must be repeated at each integration point of each finite element in each loading step. Evaluating the rates (or increments) of the logarithmic strain tensor appears also to be formidably complicated (see Hill, 1968; Gurtin and Spear, 1983; Hoger, 1987), and the computationally costly spectral decomposition must again be used (e.g., Weber and Anand, 1990).

So, from the viewpoint of data fitting, that leaves the generalization of the engineering strain as probably the best strain measure for a large-strain finite element program. This generalization is represented by the Biot strain tensor $\check{\epsilon}$. This tensor is easy to interpret because it exactly coincides with the engineering strain if there is no material rotation, i.e., if $\boldsymbol{R} = \boldsymbol{I}$. The absence of rotation is typical for most test data available for calibrating constitutive relations. The GL strain can of course be also used, but is not intuitively understandable for fitting of large strain data.

Efficient Algorithm to Compute Biot Strain

Let $\boldsymbol{X}$ and $\boldsymbol{x}$ be the initial and final coordinate vectors of material points, and let $\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$ be the displacement gradient tensor. In the explicit algorithm for dynamics, tensor $\boldsymbol{F}$ is supplied as the input to the constitutive subroutine (for which a very effective form in the case of concrete or rock is the microplane model, Bažant et al. 1996a). The GL finite strain tensor is

$$\epsilon = (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})/2 \tag{1}$$

where $\boldsymbol{I}$ is the second-rank unit tensor. Let us introduce the polar decomposition:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \tag{2}$$

where $\boldsymbol{U}$ is the right-stretch tensor and $\boldsymbol{R}$ is the rotation tensor.

The rotation tensor $\boldsymbol{R}$ may be effectively computed in finite element programs by adding its increments according to the Hughes-Winget (1980) algorithm used in ABAQUS (Hibbitt et al., 1994) or another similar algorithm by Rashid, 1993). Then $\boldsymbol{U}$ and the Biot strain tensor $\check{\epsilon}$ can be easily evaluated as

$$\boldsymbol{U} = \boldsymbol{R}^T\boldsymbol{F}, \qquad \check{\epsilon} = \boldsymbol{U} - \boldsymbol{I} \tag{3}$$

This procedure is computationally much more efficient than calculating $\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}}$ as a square root by spectral decomposition.

Since the Hughes-Winget algorithm is only approximate, a small error may nevertheless accumulate causing that $\boldsymbol{U}^2$ might not be equal to $\boldsymbol{F}^T\boldsymbol{F}$ exactly. If, after many loading steps, the difference of the norms,

$$\Delta = |\boldsymbol{F}^T\boldsymbol{F}| - |\boldsymbol{U}^2| \tag{4}$$

exceeds a certain small tolerance Δ_0, the value of $\boldsymbol{U}$ obtained from (3) needs to be improved by making a small correction $\Delta\boldsymbol{U}$. The objective is that $(\boldsymbol{U} + \Delta\boldsymbol{U})^2 = \boldsymbol{F}^T\boldsymbol{F}$ or $\boldsymbol{U}^2 + 2\boldsymbol{U}\Delta\boldsymbol{U} + (\Delta\boldsymbol{U})^2 = \boldsymbol{F}^T\boldsymbol{F}$, exactly. Here, however, the term $(\Delta\boldsymbol{U})^2$ is a second-order small correction and can be neglected. This yields for $\Delta\boldsymbol{U}$ the condition:

$$\boldsymbol{U}\,\Delta\boldsymbol{U} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{U}^2) \tag{5}$$

This represents (for three-dimensional tensors) a system of six linear equations for the components of the necessary small correction $\Delta\boldsymbol{U}$, yielding the corrected value of $\boldsymbol{U} \leftarrow \boldsymbol{U} + \Delta\boldsymbol{U}$. If Δ still does not satisfy the given tolerance, one may again substitute this corrected $\boldsymbol{U}$ into (5) and solve for new $\Delta\boldsymbol{U}$, to further improve accuracy. However, this is usually unnecessary because the convergence of the corrections is quadratic and very fast. At the same time, one should make the correction $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$.

An alternative algorithm in which the use of $\boldsymbol{R}$ is unnecessary is also possible. If the loading steps are sufficiently small, the known value of $\boldsymbol{U}$ at the beginning of the loading step can be substituted into (5), along with the current new value of $\boldsymbol{F}$. Solving (5) then yields the increment $\Delta\boldsymbol{U}$ for the current loading step. The estimate of $\boldsymbol{U}$ for the end of the current step then is $\boldsymbol{U} \leftarrow \boldsymbol{U} + \Delta\boldsymbol{U}$. This should be substituted again into (5), the solution of which then provides the correction $\Delta\boldsymbol{U}$ to the final value of $\boldsymbol{U}$ for the loading step.

Admissibility of Pairing Back-Rotated Cauchy Stress with Biot or Green-Lagrange Strains

Following Hoger (1987), Gabriel and Bathe (1995), Eterovic and Bathe (1990), Hibbitt et al. (1994) and others, the constitutive equation of isotropic hyperelastic materials may involve a non-conjugate pair of stress and strain tensors. ABAQUS (Hibbitt et al., 1995) uses in such a pair the Cauchy stress tensor. As suggested by Adley (1995), we will consider for such as pair the back-rotated Cauchy (BRC) stress tensor, which may be defined as:

$$\hat{\boldsymbol{S}} = \boldsymbol{R}^T J \boldsymbol{S} \boldsymbol{R} \tag{6}$$

Here $\boldsymbol{S}$ is the Cauchy stress tensor, and $J\boldsymbol{S}$ represents the Kirchhoff stress tensor. For reasons of consistency (and in contrast to ABAQUS), we prefer to include in (6) the volume transformation according to the Jacobian J of the deformation gradient tensor, $J = \det \boldsymbol{F}$. Consider now that, at least for a certain range of material behavior, the constitutive law may be considered as hyperelastic and defined in terms of Biot strain $\tilde{\boldsymbol{\epsilon}}$ as:

$$\hat{\boldsymbol{S}} = \boldsymbol{\phi}(\tilde{\boldsymbol{\epsilon}}) \tag{7}$$

By inversion of (6) (achieved simply by multiplying this equation by $\boldsymbol{R}$ from the left and by $\boldsymbol{R}^T$ from the right), the Cauchy (true) stress tensor is expressed from the constitutive function as:

$$\boldsymbol{S} = J^{-1}\boldsymbol{R}\hat{\boldsymbol{S}}\boldsymbol{R}^T = J^{-1}\boldsymbol{R}\boldsymbol{\phi}(\tilde{\boldsymbol{\epsilon}})\boldsymbol{R}^T \tag{8}$$

Using the well-known relation of Cauchy stress $\boldsymbol{S}$ to 2PK stress $\boldsymbol{\sigma}$ (e.g., Bažant and Cedolin, 1992, Sec. 11.2), we get

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{F}^{-1}J\boldsymbol{S}\boldsymbol{F}^{-T} = \boldsymbol{U}^{-1}\boldsymbol{R}^T J\boldsymbol{S}\boldsymbol{R}\boldsymbol{U}^{-1} = \boldsymbol{U}^{-1}JJ^{-1}\boldsymbol{R}^T\boldsymbol{R}\hat{\boldsymbol{S}}\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U}^{-1} = \boldsymbol{U}^{-1}\hat{\boldsymbol{S}}\boldsymbol{U}^{-1} \\ &= \boldsymbol{U}^{-1}\boldsymbol{\phi}(\tilde{\boldsymbol{\epsilon}})\boldsymbol{U}^{-1} \end{aligned} \tag{9}$$

in which $\boldsymbol{F}^{-T} = (\boldsymbol{F}^T)^{-1} = (\boldsymbol{F}^{-1})^T$ and the relations $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$ have been used.

The GL strain tensor may be regarded as a function of Biot strain tensor:

$$\boldsymbol{\epsilon} = \boldsymbol{H}(\tilde{\boldsymbol{\epsilon}}) = \frac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{I}) = \frac{1}{2}[(\boldsymbol{I} + \tilde{\boldsymbol{\epsilon}})^2 - \boldsymbol{I}] \tag{10}$$

where $\boldsymbol{H}$ is a tensor-valued function of a tensor (a one-to-one nonsingular mapping). The Biot stress tensor $\tilde{\boldsymbol{\sigma}}$, i.e. the stress tensor that is work-conjugate to $\tilde{\boldsymbol{\epsilon}}$, is defined by the variational relation $\boldsymbol{\sigma} : \delta\boldsymbol{\epsilon} = \tilde{\boldsymbol{\sigma}} : \delta\tilde{\boldsymbol{\epsilon}}$ where a colon (:) denotes a tensor product contracted on two indices. This relation may be written as $[\boldsymbol{\sigma} : (\partial\boldsymbol{\epsilon}/\partial\tilde{\boldsymbol{\epsilon}})] : \delta\tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\sigma}} : \delta\tilde{\boldsymbol{\epsilon}}$. Since this must hold for any variation $\delta\tilde{\boldsymbol{\epsilon}}$, it is necessary that (see also Bažant, 1995):

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} : \boldsymbol{\Psi}(\tilde{\boldsymbol{\epsilon}}), \quad \boldsymbol{\Psi}(\tilde{\boldsymbol{\epsilon}}) = \frac{\partial \boldsymbol{H}(\tilde{\boldsymbol{\epsilon}})}{\partial \tilde{\boldsymbol{\epsilon}}} \tag{11}$$

where $\boldsymbol{\Psi}$ is a fourth-rank tensor. Substitution of (9) into (11) now yields:

$$\begin{aligned} \tilde{\boldsymbol{\sigma}} &= [\boldsymbol{U}^{-1}\boldsymbol{\phi}(\tilde{\boldsymbol{\epsilon}})\boldsymbol{U}^{-1}] : \boldsymbol{\Psi}(\tilde{\boldsymbol{\epsilon}}) \\ &= [(\boldsymbol{I} + \tilde{\boldsymbol{\epsilon}})^{-1}\boldsymbol{\phi}(\tilde{\boldsymbol{\epsilon}})(\boldsymbol{I} + \tilde{\boldsymbol{\epsilon}})^{-1}] : \boldsymbol{\Psi}(\tilde{\boldsymbol{\epsilon}}) = \boldsymbol{f}(\tilde{\boldsymbol{\epsilon}}) \end{aligned} \tag{12}$$

The tensor-valued function denoted as $\boldsymbol{f}(\tilde{\boldsymbol{\epsilon}})$ has the meaning of the standard constitutive law written in terms of Biot strain and its work-conjugate stress. The important point is that the expression (12) obtained for this function on the basis of (7) is independent of rotation $\boldsymbol{R}$ (and also of the volume change characterized by J). This fact, which would not be true if the Cauchy stress tensor $\boldsymbol{S}$ instead of the rotated Cauchy stress tensor $\hat{\boldsymbol{S}}$ were used in (7), proves that (7) is a legitimate form of the constitutive law.

Conversely, if the constitutive law is given by function $\boldsymbol{f}(\tilde{\epsilon})$ in terms of work-conjugate quantities, one may deduce from it constitutive law $\boldsymbol{\phi}(\tilde{\epsilon})$:

$$\begin{aligned} \hat{\boldsymbol{S}} &= \boldsymbol{R}^T J \boldsymbol{S} \boldsymbol{R} = \boldsymbol{R}^T J(J^{-1}\boldsymbol{F}\boldsymbol{\sigma}\boldsymbol{F}^T)\boldsymbol{R} = \boldsymbol{R}^T \boldsymbol{R}\boldsymbol{U}\boldsymbol{\sigma}\boldsymbol{U}\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{U}\boldsymbol{\sigma}\boldsymbol{U} \\ &= [\boldsymbol{I} + \boldsymbol{H}(\tilde{\epsilon})]\boldsymbol{f}(\tilde{\epsilon}) : [\boldsymbol{\Psi}(\tilde{\epsilon})]^{-1}[\boldsymbol{I} + \boldsymbol{H}(\tilde{\epsilon})] = \boldsymbol{\phi}(\tilde{\epsilon}) \end{aligned} \tag{13}$$

Again it is important that, due to using $\hat{\boldsymbol{S}}$ instead of $\boldsymbol{S}$, function $\boldsymbol{\phi}$ defined by this equation is independent of $\boldsymbol{R}$. Note that this function is also independent of J.

It is now convenient to define the hyperelastic constitutive relation as a relation of the GL strain tensor to the BRC stress tensor, i.e.

$$\hat{\boldsymbol{S}} = \boldsymbol{\psi}(\boldsymbol{\epsilon}) \tag{14}$$

The Cauchy stress tensor is obtained as:

$$\boldsymbol{S} = J^{-1}\boldsymbol{R}\hat{\boldsymbol{S}}\boldsymbol{R}^T = J^{-1}\boldsymbol{R}\boldsymbol{\psi}(\boldsymbol{\epsilon})\boldsymbol{R}^T \tag{15}$$

Substituting this into (8) and calculating the 2PK stress as before, we obtain:

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{F}^{-1}J\boldsymbol{S}\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}J(J^{-1}\boldsymbol{R}\hat{\boldsymbol{S}}\boldsymbol{R}^T)\boldsymbol{F}^{-T} = \boldsymbol{U}^{-1}\hat{\boldsymbol{S}}\boldsymbol{U}^{-1} = \boldsymbol{U}^{-1}\boldsymbol{\psi}(\boldsymbol{\epsilon})\boldsymbol{U}^{-1} \\ &= (\boldsymbol{I} + 2\boldsymbol{\epsilon})^{-1/2}\boldsymbol{\psi}(\boldsymbol{\epsilon})(\boldsymbol{I} + 2\boldsymbol{\epsilon})^{-1/2} = \boldsymbol{g}(\boldsymbol{\epsilon}) \end{aligned} \tag{16}$$

This is equivalent to the standard constitutive law relating 2PK stress to GL strain. The fact that function $\boldsymbol{g}(\boldsymbol{\epsilon})$ defined by this equation is independent of rotation $\boldsymbol{R}$ (as well as J) justifies the admissibility of the constitutive law in (14). For programming, a constitutive model of the form of (14) is the simplest, but complex test data are difficult to fit in that form.

SUMMARY OF MICROPLANE MODEL AND FINITE ELEMENT APPROACH

A previous study (Bažant et al., 1996a) presented an improvement of the microplane model for concrete—a constitutive model in which the nonlinear triaxial behavior is characterized by relations between the stress and strain components on a plane, called microplane, of any orientation in the material, subject to the constraint that the strains on the microplane are the projections of the (macroscopic) strain tensor. The improvement was achieved by a new concept: the stress-strain boundaries, representing boundaries in the microplane stress-strain space which can never be exceeded. The advantage of this new concept is that various boundaries and the elastic behavior can be defined as functions of different strain variables. Thus, whereas for compression the stress-strain boundaries are defined on the microplane separately for volumetric and deviatoric components, for tension an additional boundary is defined in terms of the total normal strains. This is necessary to achieve a realistic triaxial response at large tensile strains. For microplane shear, a friction law with cohesion was introduced. The model developed was simpler than the original microplane model. An extension of the model to moderately large finite strains was also formulated.

In a subsequent study (Bažant et al., 1996b), this new microplane model was verified and calibrated by comparisons with extensive test data. A new approximate method was proposed for data delocalization, i.e., decontamination of laboratory test data afflicted by localization of strain-softening damage and size effect. This delocalization method, which is applicable to any constitutive model with strain softening, is based on a series-coupling model and on the size effect law proposed by Bažant. An effective and simplified method of material parameter identification, exploiting affinity transformations of the stress-strain curves, was also given. Only five parameters need to be adjusted if a complete set of uniaxial, biaxial and triaxial test data is available, and two of them can be determined separately in advance from the volumetric compression curve. If the data are limited, fewer parameters need to be adjusted. The parameters were formulated in such a manner that two of them represent scaling by affinity transformation. Normally only these two parameters need to be adjusted, which can be done by simple closed-form formulae. The new model allowed a very good fit of all the basic types of uniaxial, biaxial and triaxial test data for concrete. A method of nonlocal generalization was also presented.

A further study (Bažant, 1996) dealt with the finite strain generalization of small-strain constitutive equations for isotropic materials for which the strain is split into a volumetric part and a deviatoric part (the latter characterizing the isochoric strain, i.e., a strain at constant volume). The volumetric-deviatoric split had previously been handled by a multiplicative decomposition of the transformation tensor. But the existing sophisticated complex constitutive models for small strains of cohesive pressure-sensitive dilatant materials such as concrete and geomaterials involve an additive decomposition and would be difficult to convert to a multiplicative decomposition. It was shown that an additive decomposition of any finite strain tensor, and of the Green-Lagrange strain tensor in particular, is possible, provided that the higher-order terms of the deviatoric strain tensor are allowed to depend on the volumetric strain. This dependence is negligible for concrete and geomaterials because the volumetric strains are normally small whether or not the deviatoric strains are large. A method by which the stress tensor that is work-conjugate to any finite strain tensor can be converted to the Green-Lagrange strain tensor is presented. Finally, the question of the choice of the finite-strain measure to be used for the finite-strain generalization was analyzed. A transformation of the Green-Lagrange finite strain tensor whose parameters approximately reflect the degrees of freedom equivalent to replacing the small strain tensor by any other possible finite strain measure was proposed. However, the required transformation becomes too complicated for complex constitutive laws, such as that for concrete, and direct fitting of test data for finite strain might be necessary.

Acknowledgment: Partial financial support under contract 0650-350-C403 between Waterways Experiment Station (WES), Vicksburg, Mississippi, and Northwestern University (monitored by Dr. J. Zelasko) is gratefully acknowledged. Additional partial funding for finite strain tests of concrete to be reported at the Conference was obtained from the ACBM Center at Northwestern University.

REFERENCES

Adley, M.D. (1995). Private communication to Z.P. Bažant.

Hibbitt, H.D., Karlsson, B.I., and Sorensen, P. (1994). *ABAQUS Theory Manual* (1994), Version 5.4. Hibbitt, Karlsson & Sorensen, Inc., Pawtucket, Rhode island, Sec. 4.6.1.

Bažant, Z.P. (1996). "Finite-strain generalization of small-strain constitutive relations for any finite strain tensor and additive volumetric-deviatoric split." *Int. J. of Solids and Structures* 33 (20-22), 2887–2897.

Bažant, Z.P., Xiang, Y., and Prat, P.C. (1996a). "Microplane model for concrete. I. Stress-strain boundaries and finite strain." *ASCE J. of Engrg. Mechanics* 122 (3), 245-254.

Bažant, Z.P., Xiang, Y., Adley, M.D., Prat, P.C., and Akers, S.A. (1996b). "Microplane model for concrete. II. Data delocalization and verification." *ASCE J. of Engrg. Mechanics* 122 (3), 255-262.

Bažant, Z.P., and Cedolin, L. (1991). *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York.

Eterovic, A.L., and Bathe, K.J. (1990). "A hyperelastic-based large-strain elastoplastic constitutive formulation..." *Int. J. for Num. Meth. in Engrg.* 30, 1099–1114.

Gabriel, G., and Bathe, K.J. (1995). "Some computational issues in large strain elasto-plastic analysis." *Computers and Structures* 56 (2/3), 249–267.
Hill, R. (1968). "On constitutive inequalities for simple materials." *J. of the Mechanics and Physics of Solids* 16, 229–242, 315–322.
Hoger, A. (1987). "The stress conjugate to logarithmic strain." *Int. J. of Solids and Structures* 23 (12), 1645–1656.
Hughes, T.J.R., and Winget, J. (1980), "Finite rotation effects in numerical integration of rate constitutive equations arising in large-deformation analyses." *Int. J. of Numer. Methods in Engrg.* 15, 1862–1867.
Malvern, L.E. (1969). *Introduction to the mechanics of a continuous medium.* Prentice Hall, Englewood Cliffs, N.J.
Ogden, R.W. (1984). *Non-linear elastic deformations.* Ellis Horwood, Ltd., Chichester, U.K. and John Wiley and Sons, Chichester, U.K.
Rashid, M.M. (1993). "Incremental kinematics for finite element applications." *Int. J. for Num. Meth. in Engng.* 36, 3937–3956.
Weber, G., and Anand, L. (1990). "Finite deformation constitutive equations and a time integration procedure for isotropic, hyperelastic-viscoplastic solids," *Comp. Methods in Applied Mechanics and Engng.* 79, 173–202.

Appendix 1

The strain tensor $\hat{\epsilon}$ that is work-conjugate to the rotated Cauchy stress may be defined by a variational equation stating equivalence of complementary work:

$$\hat{\epsilon} : \delta \boldsymbol{S} = \boldsymbol{\epsilon} : \delta \boldsymbol{\sigma} \tag{17}$$

Substituting (6) and (9) for $\boldsymbol{S}$ and $\boldsymbol{\sigma}$, and noting that only stresses (but not deformations and rotations) are subjected to variations in the case of complementary work, we have

$$\hat{\epsilon} : \boldsymbol{R}^T \delta \boldsymbol{S}\, \boldsymbol{R} = \boldsymbol{\epsilon} : \boldsymbol{F}^{-1} \delta \boldsymbol{S}\, \boldsymbol{F}^{-T} \tag{18}$$

Introducing $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, and denoting $\delta \boldsymbol{X} = \boldsymbol{R}^T \delta \boldsymbol{S} \boldsymbol{R}$, $\boldsymbol{Y} = \boldsymbol{U}^{-1}$, we get

$$\hat{\epsilon} : \delta \boldsymbol{X} = \boldsymbol{\epsilon} : \boldsymbol{Y} \delta \boldsymbol{X} \boldsymbol{Y} \qquad \text{or} \qquad \hat{\epsilon}_{ij} \delta X_{ij} = \epsilon_{kl} Y_{ki} Y_{jl} \delta X_{ij} \tag{19}$$

To satisfy this variational equation for any δX_{ij}, it is necessary that $\hat{\epsilon}_{ij} = Y_{jl} \epsilon_{lk} Y_{ki}$ or

$$\hat{\epsilon} = \boldsymbol{U}^{-1} \boldsymbol{\epsilon} \boldsymbol{U}^{-1} = \frac{1}{2} \boldsymbol{U}^{-1} (\boldsymbol{U}^2 - \boldsymbol{I}) \boldsymbol{U}^{-1} = \frac{1}{2} (\boldsymbol{I} - \boldsymbol{U}^{-2}) \tag{20}$$

This tensor, representing the strain tensor that is work-conjugate to the rotated Cauchy stress, is now seen to coincide with the Doyle-Ericksen strain tensor

$$\boldsymbol{\epsilon}^{(m)} = (\boldsymbol{U}^m - \boldsymbol{I})/m \tag{21}$$

(Bažant and Cedolin, 1991, Sec. 11.1) for parameter value $m = -2$. This tensor is similar but not identical to Almansi strain tensor $\boldsymbol{\epsilon}_A = (\boldsymbol{I} - \boldsymbol{V}^{-2})/2 = (\boldsymbol{I} - \boldsymbol{R}\boldsymbol{U}^{-2}\boldsymbol{R}^T)/2$, in which $\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T$ = left stretch tensor, defined by polar decomposition $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$. Some authors, however, called (20) the Almansi strain tensor.

Tensor $\hat{\epsilon}$ needs to be used when the work per unit initial volume of the material needs to be calculated on the basis of the rotated Cauchy stress. Alternatively, of course, the work may be calculated on the basis of Biot stress and Biot strain, or other combinations.

By a similar calculation, it may be shown that the strain conjugate to Cauchy stress would be $J\boldsymbol{F}^{-T}\boldsymbol{\epsilon}\boldsymbol{F}^{-1}$ or $J\boldsymbol{R}\boldsymbol{U}^{-1}\boldsymbol{\epsilon}\boldsymbol{U}^{-1}\boldsymbol{R}^T$. Obviously, this strain would depend on rotation, and so it would not really be a legitimate strain measure. It is for this reason that the rotation of Cauchy stress is necessary.