Bažant, Z.P., and Novák, D. (2003). "Stochastic models for deformation and failure of quasibrittle structures: recent advances and new directions." Computational Modelling of Concrete Structures (Proc., EURO-C Conf., St. Johann im Pongau, Austria), N. Bicanic, R. de Borst, H. Mang and G. Meschke, eds., A.A. Balkema Publ., Lisse, Netherlands, 583-598.

Stochastic models for deformation and failure of quasibrittle structures: Recent advances and new directions

Zdeněk P. Bažant
Northwestern University, Evanston, Illinois 60208, USA
Drahomír Novák
Brno University of Technology, Brno, Czech Republic

ABSTRACT: The paper presents an apercu of the current problems of probabilistic failure analysis of concrete structures and proposes a new approach to finite element estimation of loads of very small failure probability. The conference presentation begins by brief comments on the problem of interplay of the nonlocal characteristic length, associated with the deterministic models for cohesive fracture or softening damage, with the autocorrelation length of the random field of local material strength, a problem on which important results were achieved in Delft by de Borst, Carmeliet & Gutiérez. The rest of the paper is focused on the problem of loads of extremely low failure probability (such as 10^{-7}), which should form the basis of rational design. For quasibrittle structures, this problem can be based on the nonlocal generalization of Weibull statistical theory. A salient aspect of the theory is the statistical-deterministic size effect. In the typical case of failures at crack initiation, the distribution tail can be checked by scaling the structure to a very large size. In the limit, the randomly located fracture process zone becomes infinitely small compared to the structure size (i.e., a point), which means that failure at one point causes the whole structure to fail. In that limit case, the tail of distribution of the failure load cannot be anything but Weibull (which follows from Fisher and Tippett's condition of form stability of the extreme value distributions). The existing stochastic finite element methods fail this fundamental requirement, which means that their far-off probability structure cannot be realistic. A new stochastic finite element formulation which guarantees the probability tail of structural strength and the large-size asymptotic size effect to be of the Weibull type is presented. A numerical example demonstrates a good representation of the statistical data of Koide et al. Furthermore, a simple formula for the mean size effect, recently derived by asymptotic matching, is discussed. Finally, a recent more fundamental derivation of the formulas for the mean size effect and for the standard deviation and entire probability distribution of the failure loads, which has been based on the nonlocal generalization of Weibull theory, is reviewed.

1 INTRODUCTION

Probabilistic mechanics is of paramount importance for progress in concrete design because the errors due to the uncertainty of the currently used empirical safety factors and reliability indices are much larger than the errors of finite element analysis. Stochastic finite element analysis has been intensely studied in the 1980s and significant progress has been achieved. However, the probabilistic structure of the existing formulations suffices only for the computation of the first and second moment statistics, that is, the means and the standard deviations. This is adequate for the design against excessive deflections and flexibility, for which probabilities between 10^{-1} and 10^{-2} are generally acceptable, but not for the very small tail probabilities required for safety against failure.

The probabilistic structure of the existing stochastic finite element method is at present not yet adequate for yielding the failure loads of extremely small probabilities such as 10^{-7}, on which the design of civil engineering structures must be based. The problem is not the numerical computation of such loads from the given statistical distributions of material properties, but the very formulation of these statistical distributions themselves. These properties must be incorporated in a way that would yield realistic far-off tails of the distributions as dictated by the known established results of the extreme value statistics (Fisher & Tippett 1928, Tippett 1925, Peirce 1925, Fréchet 1927, von Mises 1936, Gnedenko 1943, Gnedenko & Kolmogorov 1954, Epstein 1948, Freudenthal 1956,

1968, 1981, Saibel 1969, Gumbel 1954, 1958).

Concrete as well as many other materials behave in a quasibrittle manner. Their failure zone, called the fracture process zone (FPZ) is not negligibly small, as in brittle materials, but it does not behave in a plastic manner, as in brittle ductile materials. Instead, it undergoes distributed cracking, or strain-softening. When the structure is of a similar size as the FPZ, it behaves in an almost plastic manner, with no size effect, no localized failure, and no weakest-link-type randomness. When the structure is far larger than the FPZ, it fails in an almost perfectly brittle manner, and it also follows the weakest-link statistical model unless the location of fracture front is restricted by a notch or previous stable growth of a large crack. The effective size of the fracture process zone, or the characteristic length of the material, is a key parameter whose ratio to the structure size governs the brittleness or non-brittleness of response and, evidently, the size effect. Since the field of random strength must be autocorrelated, the characteristic length interacts with the autocorrelation length of the random field.

After recalling the basic facts about the cohesive, crack band and nonlocal models of quasibrittle failure (Hillerborg et al. 1976, 1983; Petersson 1961, Bažant 1976, 1984, Bažant et al. 1984, 1985, 2002, Pijaudier-Cabot & Bažant 1987, Peerlings et al. 1996, Bažant & Planas 1998), Weibull (1939) theory of perfectly brittle failure, and size effect theories (Bažant 1987, 1999, 2001; Bažant & Chen 1987), the conference presentation will briefly review statistical generalization of nonlocal finite element models for softening damage and especially the recent studies in Delft (see Appendix I) clarifying the interplay of the characteristic length for nonlocal analysis and the autocorrrelation length of the random local strength field. The nonlocal generalization of Weibull statistical theory of brittle failure will then be discussed and the limitations of the statistical size effect identified. The rationale behind the amalgamated energetic-statistical size effect law will be examined and the underlying asymptotic matching argument discussed. A new derivation of this size effect from the probabilistic nonlocal theory will be outlined. Finally, a generalization of nonlocal finite element analysis to extreme value statistics will be proposed and demonstrated by numerical examples.

2 WEIBULL THEORY OF RANDOM STRENGTH AND ITS LIMITATIONS

The Weibull statistical theory of random strength and size effect is based on the weakest link model for the failure of a chain consisting of links whose strengths are statistically independent random variables (Fig. 1b). In many two- or three-dimensional structures, the failure occurs as soon as one small element of material (representative volume V_r) fails. From the probabilistic viewpoint, such structures are analogous to the weakest link model for a chain. In the continuum limit of infinitely many infinitely small links, the weakest link model leads to the following cumulative distribution of failure probability:

$$P_f(\sigma_N) = 1 - e^{-\int_V c[\boldsymbol{\sigma}(\boldsymbol{x},\sigma_N)]\mathrm{d}V(\boldsymbol{x})} \quad (1)$$

$$c(\boldsymbol{\sigma}) = \sum_{I=1}^{3} \frac{P_1[\sigma_I(\boldsymbol{x})]}{V_r} \quad (2)$$

where $\sigma_I(\boldsymbol{x})$ = principal stress just before failure at point of coordinate vector $\boldsymbol{x}$ ($I = 1, 2, 3$), V = volume of structure, and $c(\boldsymbol{\sigma})$ = function giving the spatial concentration of failure probability of the material (= $V_r^{-1}\times$ failure probability of material representative volume V_r) (Freudenthal 1968, Bažant & Planas 1998); $c(\boldsymbol{\sigma})$ = concentration function (spatial density of failure probability); and $P_1(\sigma_I)$ = failure probability (cumulative) of the smallest possible test specimen, of volume V_r, subjected to stress σ_I. Eq. (1) gives the failure probability of the structure, provided that the structure (with the loading system) is of such a geometry that the failure (the maximum load) occurs as soon as a macroscopic crack initiates (this is called positive geometry—a geometry for which the energy release function increases with the crack length). Eq. (1) can be derived by noting that the survival probability, $1 - P_f$, of a chain of N links is the joint probability that all the links survive; this implies that $1 - P_f = (1 - P_1)^N$. Of practical interest are only very small failure probabilities P_1 and P_f, and so we may write $\ln(1 - P_f) = N \ln(1 - P_1) \approx -NP_1$ or $1 - P_f = \mathrm{e}^{-NP_1}$.

As proven mathematically by Fisher & Tippett (1928) and also justified by Freudenthal on physical grounds (based on an analysis of material flaws), the low-probability tail of $P_1(\sigma)$ must be a power law:

$$P_1(\sigma) = \left\langle \frac{\sigma - \sigma_u}{s_0} \right\rangle^m \quad (3)$$

(Weibull 1939) where m, s_0, σ_u = material constants (m = Weibull modulus, usually between 5 and 60). The threshold σ_u cannot be negative and is typically taken as 0 (because it is next to impossible to identify σ_u from the available test data unambiguously since different positive σ_u can give almost equally good fits of data). Substitution of 3 into Eq. (1) yields what came to be known as the Weibull distribution. This distribution (for $\sigma_u = 0$) to simple expressions for the mean of σ_N as a function of m and the coefficient of

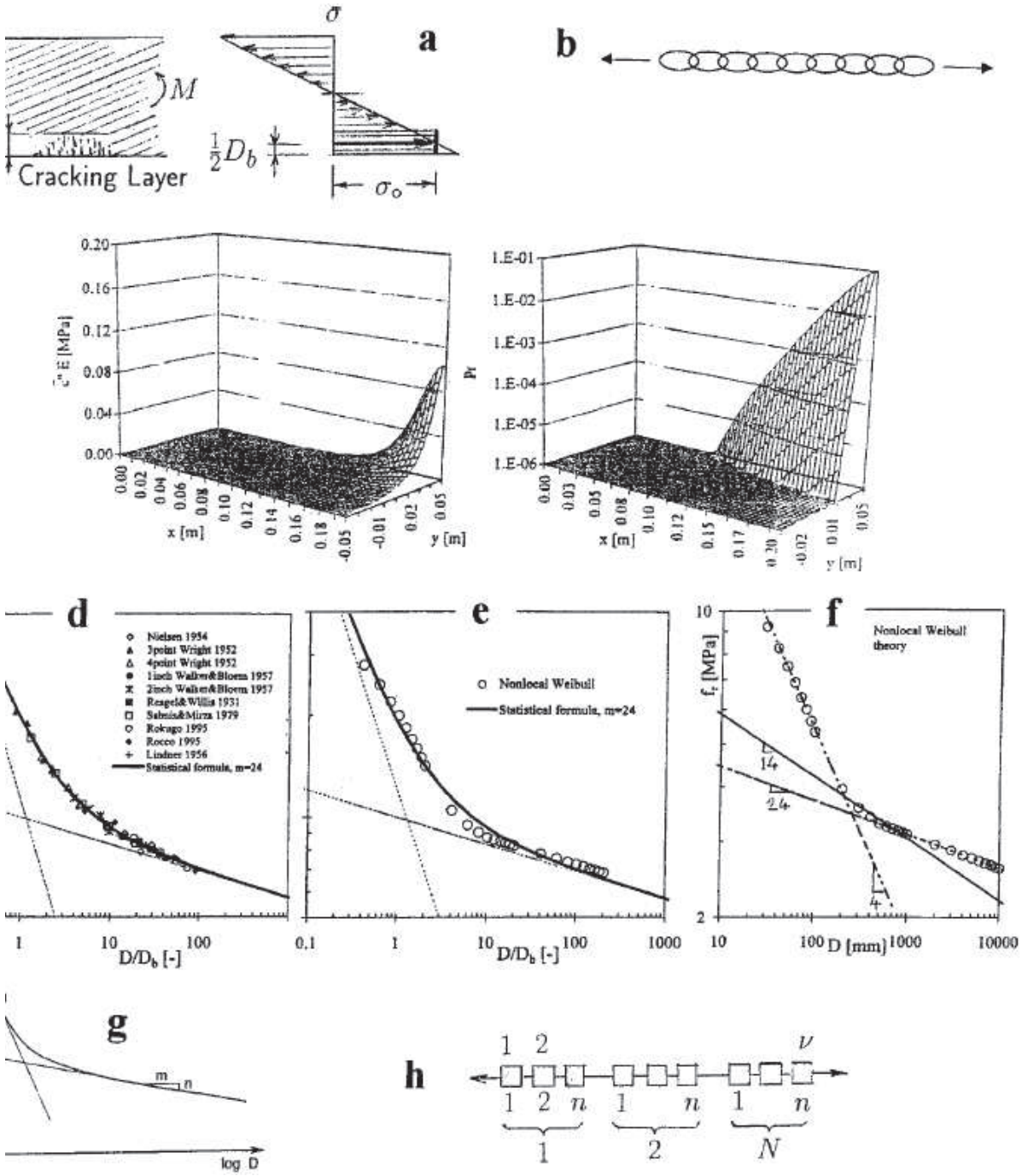

Figure 1: (a) Approximate effect of stress redistribution due to cracking in boundary layer. (b) Chain structure. (c) Field of inelastic strain in beam flexure (linear scale) and corresponding density field of contributions to failure probability (log-scale). (d,e) Relative flexural strength versus relative size for 10 test series from the literature, and results of nonlocal Weibull calculations, fit by energetic-statistical formula for crack initiation. (f) Differences in apparent Weibull moduli m corresponding to nonlocal Weibull calculations in different size ranges. (g) Energetic-statistical size effect for failures at crack initiation. (h) Chain subdivided into segments.

variation ω of σ_N,

$$\sigma_N = s_0\,\Gamma\!\left(1+\frac{1}{m}\right)\left(\frac{V_r}{V}\right)^{1/m} \propto D^{-n_d/m} \quad (4)$$

$$\omega = \left(\frac{\Gamma(1+2m^{-1})}{\Gamma^2(1+m^{-1})}-1\right)^{1/2} \quad (5)$$

where Γ is the gamma function, and n_d = 1, 2 or 3 for uni-, two- or three-dimensional similarity. Eq. (4) represents a power-law size effect on the mean nominal strength $\overline{\sigma}_N$ (if $\sigma_u = 0$). Since there is no size effect on ω, the expression for ω in (4) is normally used to identify m from tests.

Eq. (4) for $\overline{\sigma}_N$ suggests adopting the value $\sigma_W = \sigma_N(V/V_0)^{1/m}$ for a uniformly stressed specimen as a size-independent stress measure. Considering from this viewpoint a large crack-tip plastic zone in metals, Beremin (1983) proposed the idea of the so-called Weibull stress:

$$\sigma_W = (\sum_k {\sigma_j}^m\, V_j/V_r\,)^{1/m} \quad (6)$$

where V_j ($j = 1, 2, ...N_W$) are the elements of the plastic zone having maximum principal stress σ_{Ij}. Ruggieri & Dodds (1996) replaced the sum in (6) by an integral (see also Lei et al. 1998). However, Eq. (6) can be applied only if the crack at the moment of failure is still microscopic, that is, small compared to the structural dimensions. Therefore, the concept of Weibull stress is not useful the case for quasibrittle materials in which the process zone can be as large as the structure.

When the structure geometry is not positive (which for example occurs in beams with tensile reinforcement, or when an adjacent compressed zone stabilizes a crack, as in gravity dams), a large crack must develop before the failure can occur. This precludes Weibull-type statistical analysis. Although rigorous probabilistic modeling seems prohibitively difficult, the case of negative geometry is not very important because the size effect is predominantly energetic (deterministic). So, when the size effect is mainly statistical, the violations of statistical independence of material elements at different locations have a negligible effect, and when it is not, the question of statistical independence of these elements becomes irrelevant.

In the case of quasibrittle structures, applications of the classical Weibull theory face a number of fundamental objections:

1) The fact that the size effect on σ_N is a power law means that the functional equation (1) is satisfied, and this implies the absence of any characteristic length. But this cannot be true if the material does contain sizable inhomogeneities, as does concrete.

2) The energy release due to stress redistributions caused by a macroscopic FPZ or a stable crack growth before P_{max} gives rise to a deterministic size effect, which is ignored. Thus the Weibull theory can be valid only if the structure fails as soon as a microscopic crack becomes macroscopic.

3) Every structure is mathematically equivalent to a uniaxially stressed chain or bar of a variable cross section, which means that the structural geometry and failure mechanism are ignored.

4) The size effect differences between the cases of two- and three-dimensional similarities (n_d = 2 or 3) are often much smaller than predicted by Weibull theory (because, for example, a crack in a beam causes failure only if it spreads across the full width of the beam).

5) Many tests of quasibrittle structures show a much stronger size effect than predicted by Weibull theory (e.g., diagonal shear failure of reinforced concrete beams; Walraven & Lehwalter 1994, Walraven 1995, Iguro et al. 1985, Shioya & Akiyama 1994, and many flexure tests of plain beams cited in Bažant & Novák 2000a,b).

6) When Weibull exponent m is identified by fitting the standard deviation of σ_N for specimens of very different sizes, very different m values are obtained. Also, the size effect data and the standard deviation data give very different m (e.g., $m = 12$ was obtained with small concrete specimens while the large-size asymptotic behavior corresponds to $m = 24$ (Bažant and Novák 2000); Fig. 1f (m varies from 4.2 to 24.2).

7) The classical theory neglects spatial correlations of material failure probabilities (which is admissible only if the structure is far larger than the autocorrelation length l_a of the random field of local material strength).

3 OVERVIEW OF NONLOCAL GENERALIZATION OF WEIBULL THEORY

One can discern three approaches to generalizing the Weibull theory in various ways, and to various degrees, the capture the effect of a large FPZ and quasibrittleness.

1) One classical approach is represented by various phenomenological models for load sharing (or parallel coupling of links) (Daniels 1945, Grigoriu 1990). Although these generalizations can simulate some effects of a large FPZ, they are not generally applicable and cannot capture the effect of structure geometry. Calibrating the model for one structure geometry, one cannot predict the behavior for another geometry.

2) Another classical approach attempts to overcome the problem of LEFM crack-tip singularity, which causes the classical Weibull integral to di-

verge for $m > 4$ (this includes all realistic m values); Beremin (1983), Ritchie, Becker, Lei et al. (1998), Lin, Evans, McClintock, Phoenix (1978), etc.. For example, one can exclude from the domain of Weibull integral a finite circular zone surrounding the crack tip, in order to make the integral convergent, or one can consider plastic blunting of the stress profile ahead of the crack tip, or one can average the failure probability spatially. These approaches have been shown useful for tough metals with a moderately large yielding zone at the crack tip, but are doubtful when the effective FPZ length is of the same order of magnitude as the structure size (which is typical for reinforced concrete) and are not completely general (e.g., they cannot be applicable for crack initiation from a smooth surface).

3) The most recent approach is the nonlocal Weibull theory (Bažant & Xi 1991, Bažant & Novák 2000a,b). This is a general theory which has as its limit cases *both* the classical Weibull theory and the deterministic nonlocal continuum damage mechanics developed for finite element analysis of quasibrittle materials. The energetic size effect is implied by this theory as the asymptotic case for not too large structure sizes.

The nonlocal concept was proposed for elasticity in the 1960s (Kröner 1961, Eringen 1965, Kunin, Edelen) and later extended by Eringen et al. to hardening plasticity. In the 1980s, it was adapted to strain-softening continuum damage mechanics and strain-softening plasticity (Bažant 1984. Bažant et al. 1984, Pijaudier-Cabot & Bažant 1987), with three motivations:

(1) to serve as a computational 'trick' (localization limiter) eliminating spurious mesh sensitivity and incorrect convergence of finite element simulations of damage;

(2) to reflect the physical causes of nonlocality, which are: (a) material heterogeneity, (b) energy release due to microcrack formation, and (c) microcrack interactions; and

(3) to simulate the experimentally observed size effects that are stronger than those explicable by Weibull theory.

Because of material heterogeneity, the macroscopic continuum stress at a point material must depend mainly on the average deformation of a representative volume of the material surrounding that point, rather than on the local stress or strain at that point.

In the deterministic nonlocal theory for strain softening damage or plasticity, the spatial averaging must be applied only to the inelastic part $\boldsymbol{\epsilon}''$ of the total strain $\boldsymbol{\epsilon}$ (or some of its parameters), rather than to the total strain itself (Pijaudier-Cabot & Bažant 1987). Accordinly, the cumulative failure probability $P_1(\boldsymbol{\sigma})$, considered in the classical Weibull theory as a function of the local stress tensor $\boldsymbol{\sigma}$ at continuum point $\boldsymbol{x}$, must be replaced by a function of a nonlocal variable (Bažant & Xi, 1991, Bažant & Novák 2000a,b). The nonlocal stress is not acceptable because it decreases with increasing average strain. A suitable nonlocal variable is the nonlocal strain or, more precisely, the nonlocal inelastic part of strain. The material failure probability is thus defined in the nonlocal Weibull theory as

$$P_1 = \langle \bar{\boldsymbol{\sigma}}/s_0 \rangle^m, \;\; \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = \boldsymbol{E} : [\boldsymbol{\epsilon}(\boldsymbol{x}) - \bar{\boldsymbol{\epsilon}}''(\boldsymbol{x})] \quad (7)$$

$$\bar{\boldsymbol{\epsilon}}''(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{s}-\boldsymbol{x})\boldsymbol{\epsilon}''(\boldsymbol{s})\mathrm{d}V(\boldsymbol{s}) \,/\, \bar{\alpha}(\boldsymbol{x}) \quad (8)$$

in which $\boldsymbol{E}$ = initial elastic moduli tensor; $\alpha(\boldsymbol{s}-\boldsymbol{x})$ = a bell-shaped nonlocal weight function whose effective spread is characterized by characteristic (material) length l_0; and $\bar{\alpha}(\boldsymbol{x})$ = normalizing factor of $\alpha(\boldsymbol{s}-\boldsymbol{x})$.

The nonlocality makes the Weibull integral over a body with crack tip singularity convergent for any value of Weibull modulus m, and it also introduces into the Weibull theory spatial statistical correlation. Numerical calculations of bodies with large cracks or notches showed that the randomness of material strength is almost irrelevant for the size effect on the mean σ_N, except theoretically for structures extrapolated to sizes less then the inhomogeneity size in the material (Bažant & Xi 1991). Therefore, the energetic mean size effect law for the case of large cracks or large notches remains unaffected by material randomness. Intuitively, the reason is that a significant contribution to Weibull integral comes only from the FPZ, the size of which remains constant if the structure size is increased. The same reason applies to the boundary layer of cracking (Fig. 1a), and is documented by the inelastic strain field in (Fig. 1c left, linear scale) and the field of the density of contribution to the Weibull integral (right, log-scale) obtained by Bažant & Novák (2000a) in nonlocal beam flexure analysis.

A special case in which the statistical size effect is important is the failure at crack initiation in a very large structure, much larger than the inhomogeneity size. This is the case of bending of very thick plain concrete beams or plates, for example the flexural failure of an arch dam about 10 m thick (Bažant & Novák 2000a,b; Fig. 1d,e).

The asymptotic limits of the mean nonlocal Weibull size effect are, for $D \to 0$, the deterministic energetic size effect and, for $D \to \infty$, the mean classical Weibull size effect. Their asymptotic matching approximation leads to the following approximate formula for the mean size effect(Bažant 2001, Fig. 1

g,d,e):

$$\sigma_N = \sigma_0 \left[\left(\frac{D_b}{\eta D_b + D} \right)^{rn/m} + \frac{r D_b}{\eta D_b + D} \right]^{1/r} \quad (9)$$

where $rn/m < 1$; η, r = empirical constants. The special case for $\eta = 0$ was shown to fit the bulk of the existing test data on the modulus of rupture and closely agree with numerical predictions of the nonlocal Weibull theory over the size range 1:1000 (Bažant & Novák 2000b). Aside from the two aforementioned asymptotic limits, the formula also satisfies, as a third asymptotic condition, the requirement that the deterministic size effect on the modulus of rupture must be recovered for $m \to \infty$.

As for the coefficient of variation ω_N of the nominal strength σ_N, it can be proven analytically in general that, according to the nonlocal generalization of Weibull theory, it is independent of size D. In other words, ω_N exhibits no statistical size effect, is defined by the same expression as in the classical Weibull theory (Eq. 4), and is fully determined by the value of Weibull modulus m. Numerically this fact was demonstrated in Bažant & Novák (2000a).

4 GENERALIZATION OF STOCHASTIC FINITE ELEMENT METHOD (SFEM) FOR VERY LOW PROBABILITY FAILURES

In comparison to SFEM, the nonlocal generalization of Weibull statistical theory (Bažant & Xi, 1991; Bažant & Novák 2000a,b; Bažant 2002a,b) has two limitations:

1) It does not yield the statistics of stiffness, deflections and stresses during the loading process; and

2) the failure probability is not related to the probability that the first eigenvalue λ_1 of the tangential stiffness matrix K_t of the structure, K_t, becomes nonpositive. Properly, matrix K_t should loose positive definiteness at the onset of failure.

SFEM has become a powerful tool for calculating the statistics of deflections and stresses of arbitrary structures (e.g., Schuëller 1997a,b, Kleiber & Hien 1992, Ghanem & Spanos 1991, Liu et al. 1987, Deodatis & Shinozuka 1991, Shinozuka & Deodatis 1988, Takada 1990). A critical appraisal, however, suggests that SFEM, as it now exists, is not an adequate tool for determining failure loads of very small probability, on which structural design must be based. In this regard, the nonlocal generalization of Weibull statistical theory has three important advantages:

1. In the limit of infinite size, the nonlocal Weibull theory reduces to the classical (local) Weibull theory, which is not true for the SFEMs in their contemporary form.

2. The nonlocal characteristic length ℓ, in contrast to autocorrelation length in SFEMs L_a, has a clear physical meaning and can be easily evaluated from the size effect tests using simple LEFM-based formulas; it ensues as the transitional size D_0 obtained as the intersection of the asymptotes of an optimally matched size effect law, times a shape factor known from LEFM.

3. The nonlocal Weibull theory is simpler, since an autocorrelated random field is not needed (a certain kind of spatial correlation is implied by the characteristic length of the nonlocal averaging operator).

The first point is a fundamental one. It is related to the far-out tail of the probability distribution of the tangential stiffness, which governs the failure loads of very small probability. The following physical argument is pertinent in this regard (Bažant 2001):

Imagining the structure to be scaled up to infinity size ($D \to \infty$), the FPZ becomes infinitely small compared to the structure size D (i.e., a point in the dimensionless coordinates $\boldsymbol{\xi} = \boldsymbol{x}/D$). In that case, failure (of a structure of positive geometry) must occur right at fracture initiation. Therefore, the classical (local) Weibull theory must apply, and the failure load of very low probability then depends *only* on the far-off tail of the local strength distribution at a point of the structure. Thus, extrapolation to very large sizes is a way to identify the far-off tail of the local strength distribution.

On the other hand, the existing SFEMs have not been shown to converge to Weibull theory and to reproduce the Weibull size effect as $D \to \infty$, neither analytically nor computationally. This important requirement is not ensured by the existing SFEMs.

The requirement that the Weibull size effect must be approach for infinite size has some implications for the structure of the tail of the probability distribution of the stiffness coefficients, deflections and stresses. Normally, the material stiffness characteristics are assumed in SFEM to have the Gaussian or log-normal distributions. Since the probability distribution of the structural tangential stiffness matrix is essentially a weighted sum of the elemental distributions (i.e., the distributions of the stiffness characteristics of a small representative volume of the material), the distribution of the structural stiffness coefficients may be expected to be Gaussian, with the exception (1) of the far-out tail of probability distribution, and (2) of the states of damage localization in one or several finite elements which may (though need not) occur just before reaching the peak load and dominates the structural stiffness.

During the loading process, the maximum load (which represents a failure state under the conditions

of load control), is reached when the tangential stiffness matrix of the structure, $\boldsymbol{K}_t$, ceases being positive definite, i.e., when the first eigenvalue λ_1 of $\boldsymbol{K}_t$ ceases being positive (e.g. Bažant & Cedolin 1991, ch. 4, 10 and 13). Therefore,

$$\text{Failure probability}(u) \;=\; \text{Prob}\,(\,\lambda_1(u) \leq 0\,) \qquad (10)$$

As indicated here, the failure probability and the first eigenvalue are regarded as functions of the load-point displacement u (since, in order to achieve computational stability, it is u, rather than P, which needs to be controlled during loading).

Now it is important to realize that, in the limit of infinite size, the distribution of extreme values is not arbitrary, not something that can be left to empirical observations. Rather, it must be the Weibull distribution, exactly, and nothing else. Consequently, it necessary to satisfy, in the limit of infinite size, the following condition:

$$\text{Tail of}[\,\text{Prob}\,(\,\lambda_1(u) \leq 0\,)] \;=\; F_W[P(u)] \qquad (11)$$

Here $F_W(P)$ is the cumulative Weibull distribution function (Weibull 1939). This distribution has a power-law tail and a threshold, which can normally be taken as zero (excluding negative values). F_W is here properly considered as an implicit function of the controlled displacement u because the tangential stiffness in the direction of loading changes its sign at maximum load.

Imposing condition (11) can be justified by ruling out all the other possibilities, which are as follows. In a population of N statistically independent random variables X_i ($i = 1, 2, ...N$) with arbitrary but identical statistical distributions Prob$(X_i \leq x) = P_1(x)$, henceforth called the elemental distribution ($x = \sigma/s_0$ = scaled stress, X_i = scaled random strength), the distribution of Y_N = min$_{i=1}^N$ X_i for very large N has the general expression:

$$P_N(y) = 1 - e^{-NP_1(y)} \qquad (12)$$

where $P_N(y) =$ Prob$(\min_{i=1}^N X_i \;\leq\; y)$; $P_N(y) = P_f$ = failure probability of structure, provided that the failure of one element causes the whole structure to fail. As proven by Fisher and Tippett (1928), there exist three and only three asymptotic forms (or limiting forms for $N \to \infty$) of the extreme value distribution $P_N(y)$:

1) Fisher-Tippett-Gumbel distribution:

$$P_N(y) = 1 - e^{-e^y} \qquad (13)$$

2) Fréchet distribution:

$$P_N(y) = 1 - e^{|y|^{-m}} \qquad (14)$$

3) Weibull distribution:

$$P_N(y) = 1 - e^{-y^m} \qquad (15)$$

(Case 1 is usually called the Gumbel distribution, but Fisher and Tippett derived it much earlier and Gumbel gave them credit for it.) Case 3 is obtained if the elemental distribution $P_1(y)$ has a power-law tail with a finite threshold (the simplest case is the rectangular probability density function, for which $m = 1$). Case 1 is obtained if $P_1(y)$ has an infinite exponentially decaying tail, and case 2 if $P_1(y)$ has an infinite tail with an inverse power law (such as $|\sigma|^{-m}$) (see also Bouchaud & Potters 2000).

Fisher & Tippett (1928) based their proof on three arguments: (1) The key idea is that the extreme of a sample of $\nu = Nn$ independent identical random variables x (the strengths of the individual links of a chain) can be regarded as the extreme of the set of N extremes of the subsets of n variables, e.g., the strengths of n links of a chain (Fig. 2g). (2) As both $n \to \infty$ and $N \to \infty$, the distributions of the extremes of samples of sizes n and Nn must have a similar form if an asymptotic form exists. This implies that that these distribution must be related by a linear transformation in which only the mean and the standard deviation can change; i.e., $\sigma' = a_N\sigma + b_N$ where a_N and b_N are functions of N ($N \sim$ structure size) Although an asymptotic distribution of the extremes, as a limit for $N \to \infty$, does not exist, an asymptotic *form* (or shape) of the extreme value distribution should exist, i.e., the asymptotic distribution form should be stable with regard to increasing N. Thus the argument of a joint probability of survival of all N segments of the chain yields for the asymptotic form of the cumulative distribution of the survival probability $F(\sigma) = 1 - P_f = 1 - P_N$ of a very long chain the recursive functional equation:

$$F^N(\sigma) = F(a_N\sigma + b_N) \qquad (16)$$

which is called the stability postulate of extreme value distribution (a_N, b_N are coefficients depending on N). Fisher and Tippett proved that this functional equation for unknown function F has three and only three types of solution, and that they are given by (13)–(15). By substituting these forms into functional equation (16), one can check that indeed this equation is satisfied. The substitutions further give the dependence of a_N and b_N on N, which in turn characterizes the dependence of the mean and the standard deviation of each asymptotic distribution on N ($N \sim$ structure size).

The infinite negative tails of P_N of the Fréchet distribution and the Fisher-Tippett-Gumbel distribution are not acceptable for describing the strength. Therefore, these two distributions are are ruled out. So, in

the case of strength, there is no other acceptable tail distribution but Weibull distribution (which is what lends support to the condition (11)).

In SFEMs, there are of course techniques, such as the importance sampling, for calculating the failure loads of extremely small probability. Unfortunately, though, the fact that the probability structure of the existing SFEMs do not yield Weibull power-law tails, nor lead to the Weibull power-law size effect when the structure size is scaled up to infinity, means that the calculations of loads of a very small failure probability, such as 10^{-7} (and probably not even 10^{-3}), are unrealistic.

In design codes, the safety factors relate the mean failure load prediction (roughly the same as the median, or failure probability 0.5) to the failure load with a desired extremely low probability, typically about 10^{-7}. This is illustrated by the upper arc in Fig. 2k, spanning about 6.5 orders of magnitude. Since the existing experimental validations of SFEMs have been confined mainly to the standard deviation, the current SFEMs, with their exponential tails, might be realistic for calculating only loads of failure probability no less than about 10^{-2}. So, an empirical safety factor spanning 5 orders of magnitude is needed to relate this load to the failure load of the desired probability such as 10^{-7} (see the lower arc in the figure). Comparing the lengths of the two arcs, one gets the sobering impression that, in terms of safety against failure, not too much is gained by the use of SFEMs if a proper tail probability structure is not enforced.

5 PRACTICAL FINITE ELEMENT APPROACH

In a primitive approach, the structure would have to be subdivided a very large number ν of finite elements having the fixed size of the characteristic volume of material. For a very large structure, ν would be a very large number, easily one billion. Therefore, such a primitive approach is impossible.

To achieve a proper tail probability with a manageable number of finite elements, the following idea, which is advanced in more detail in Novák, Bažant & Vořechovský (2003), will now be briefly outlined. The idea is to simulate by finite elements directly Fisher & Tippett's (1928) fundamental stability postulate of extreme value distributions. This can be done as follows.

Similar to the crack band model (Bažant 1982, Bažant & Oh 1983), the finite elements used in computations have a fixed size in the relative coordinate $\boldsymbol{\xi} = \boldsymbol{x}/D$. The structure is subdivided into a fixed number N of finite elements (called the macroelements) the size of which increases in proportion to structure size D. Each macroelement is imagined to consist of n microelements of a fixed actual size and a strength represented by an independent stochastic variable. The subdivision number is defined as $n = V_{el}/l^{n_d}$ where V_{el} is the macroelement volume, l^{n_d} is the characteristic volume of the material, l = characteristic length and n_d = number of spatial dimensions in which the structure is scaled up (n_d = 1,2,3 for one-, two- and three-dimensional similarity). Then the strength for each of these N subsets of n stochastic strength variables is simulated statistically, according to Weibull distribution, and for each subset the extreme (minimum strength) is selected to be the representative statistical property (the strength) of the finite element (macroelement). These N extremes of the subsets of n variables are then used in numerical finite element analysis of the whole structure.

This procedure ensures that the extreme value statistics is correctly approached, with one crucial advantage—the number n of finite elements (macroelements) remains reasonable from the computational point of view. Although N increases with the structure size, the determination of the extreme from the subdivision of each macroelement does not add to the computational burden since it is carried out analytically, outside the finite element analysis.

As already pointed out, one basic hypothesis of the classical Weibull theory of structural strength is the statistical independence of the strengths of the individual characteristic volumes l^{n_d} , where l is the characteristic length. The strength of each of these volumes can be described by Weibull distribution with Weibull modulus m and scale parameter s_0 (the threshold being taken as zero, as usual). Each of the aforementioned macroelements, whose characteristic size is L_0 and characteristic volume $L_0^{n_d}$, may be imagined to be discretized into n characteristic volumes l^{n_d}. This consideration provides the statistical properties of the macroelement. Since we are interested only in very small tail probabilities, we may simply substitute in these equations the tail approximation of the elemental (generic) Weibull distribution with a certain modulus and scale parameter. The tail approximation is the power function σ^m (times a constant), and its substitution leads for the strength of the macroelement again to Weibull distribution but with a different modulus and scale parameter, and thus with a different mean and variance, which are expressed according to Eq. (4).

6 NUMERICAL EXAMPLE OF FOUR-POINT-BEND FLEXURAL TESTS

By this time, abundant experimental evidence on the statistical size effect on plain concrete beams has been accumulated. Koide et al. (1998, 2000) recently reported tests of 279 plain concrete beams under four-point bending, aimed at determining the influence of

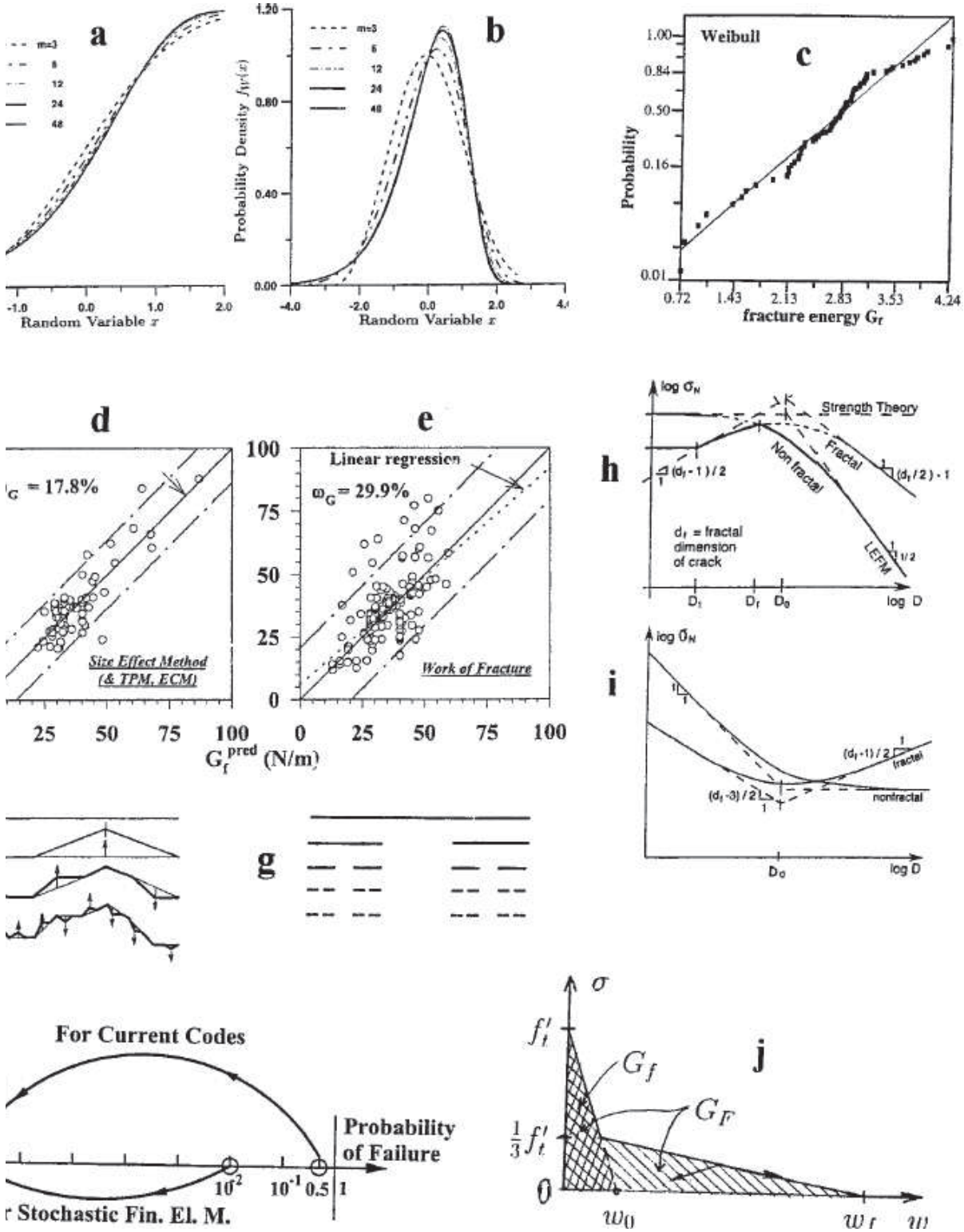

Figure 2: (a,b) Cumulative Weibull distribution and its density, for various Weibull moduli m. (c) 77 size effect test data on fracture energy from the literature, plotted on Weibull probability paper. (d,e) Plots of measured versus predicted values of size effect fracture energy (77 test series) and Hillerborg fracture energy (161 test series). (f,g) Invasive affine fractality of crack surface and lacunar fractality of microcracks. (h,i) Unreasonable size effects for large cracks and crack initiation ensuing from the hypothesis of fractal size effect. (j) Softening stress-crack separation curve of cohesive crack model of concrete. (k) Safety factors relating failure probability to calculations.

the beam length L on the flexural strength of beams. These excellent statistical data permit a comparison of the cumulative probability distribution function of the maximum bending moment M_{max} at failure (Bažant & Novák 2000b, Novák et al. 2001). Beams of three different bending spans, 200, 400 and 600 mm (series C of Koide et al.) are shown in Figure 3, along with the cracks obtained by deterministic finite element calculations (carried out with the code ATENA, Červenka 2002). The cross-sections of all the beams were kept constant (0.1m $\times$0.1m) and the experiments showed how M_{max} decreases with an increasing span (Fig. 4). To describe the size effect of the span, Koide et al. proposed a modification of the Weibull theory.

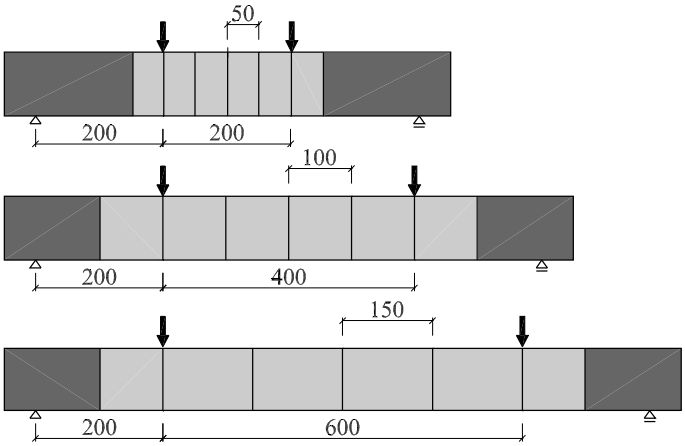

Figure 3: Koide's beams of bending span 200, 400 and 600 mm, series C.

Koide et al. reported the compression strength of the concrete, but unfortunately not the direct tensile strength and the fracture energy of concrete. The experimental results plotted in Fig. 4 give the mean value for each size. The double logarithmic plot of M_{max} versus the span is approximately a straight line of slope $D^{-n_d/m}$, where n_d is the spatial dimension and m is the Weibull modulus. Since the depth and width of the beam are not increased, the problem is properly analyzed as one-dimensional, and then the overall slope of the experimental data in the figure is matched best with the value $m = 8$, which is unusually low for concrete and implies an unusually high coefficient of variation of the scatter of flexural strength.

Deterministic simulation with nonlinear fracture mechanics software (made with ATENA) indicate that no appreciable size effect is present. This is no surprise. According to fracture mechanics, the deterministic size effect on flexural strength of beams, whether unnotched or notched, is almost nil if the beam depth is not varied because the energy release function is almost independent of the beam span. This is useful in view of our focus on the statistical size effect. It allows a purely statistical analysis of the test data in

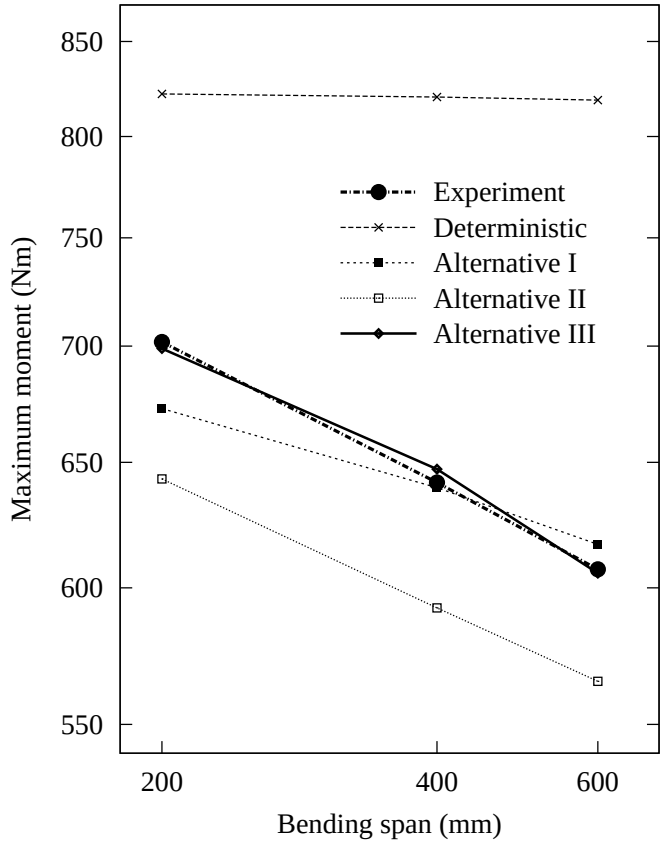

Figure 4: Comparison of means of Koide's data and ATENA Deterministic and Statistical simulations.

Fig. 4, reflecting the fact that, the longer the beam, the higher is the probability of encountering in it a material element of a given low strength.

The force increments applied on the beams were prescribed in numerical simulation in order to avoid a nonsymmetrical bending moment distribution when material randomness causes the crack pattern (Fig. 5) to become nonsymmetric. Because of load control, the load-deflection curves, including the peak and postpeak response, were calculated using the arc length method.

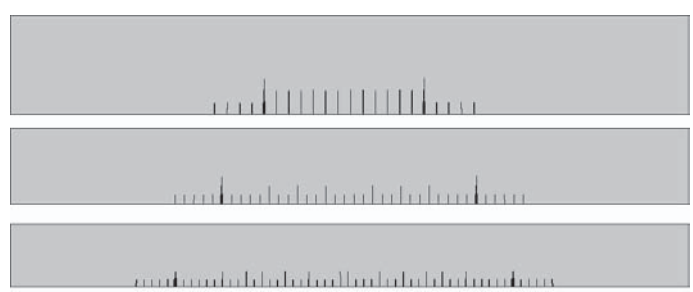

Figure 5: Deterministic cracks for sizes 20, 40 and 60.

The probabilistic version of nonlinear fracture mechanics software ATENA (Červenka & Pukl 2002) was utilized to simulate the tests of Koide et al. by finite elements in the sense of extreme value statistics. This was made possible by integrating ATENA with the probabilistic software FREET (Novák et al. 2002, 2003). In this simulation, the finite element mesh is

defined by using only 6 stochastic macroelements of strip-like form, placed in the central region of the test beams in which fracture initiates randomly; see Figure 6. The strips suffice for simulating the Weibull size effect. We imagine n elements per macroelement of width L_0, while the finite element meshes for all the sizes are identical (except for scaling by a horizontal stretch).

The characteristic length is considered to be 50 mm, which is approximately three-times the maximum aggregate size. The Weibull modulus is taken as m=8, and the scale parameter is s_0 = 1.0 MPa. The statistical parameters of the strength of the macroelements, imagined to consist of $N = L_0/l$ material elements each. For the three sizes (spans) considered here, $M = 50, 100, 150$ mm and $N = 1, 2, 3$.

For the stochastic finite element simulations, a stochastic computational model with N=6 random tensile strength variables is defined for each beam size (span). These 6 variable are characterized by 16 random simulations based on the method of Latin hypercube sampling, using simulations by the FREET and ATENA softwares (Novák et al. 2003, Vořechovský & Novák 2003, Pukl et al. 2003). The statistical characteristics of the ultimate load can then be evaluated. The mean values of nominal strength obtained from a statistical set of the maximum load are determined first. The random cracking pattern at failure is shown in Fig. 6, as obtained for four realizations of three progressively improved alternatives of solution. To illustrate the randomness of failure, the corresponding random load-deflection curves are shown in Figure 7. The three alternatives, for which the results are presented in Figure 4, are the following:

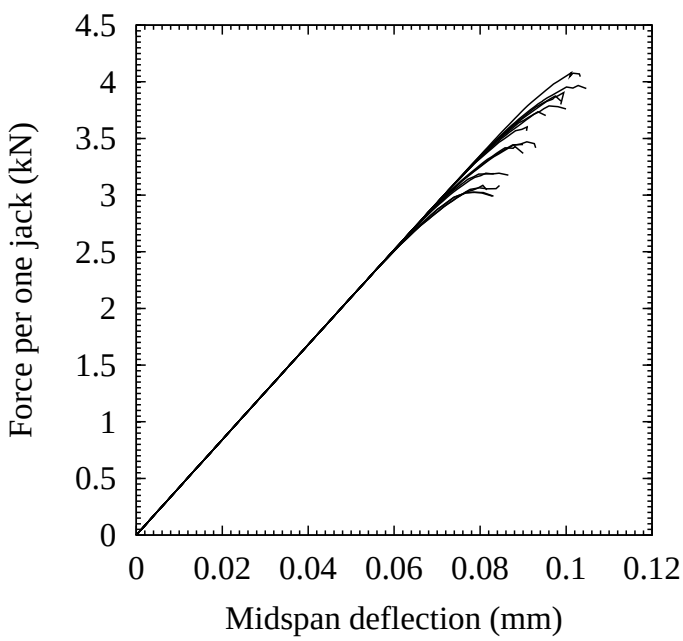

Figure 7: Random load deflection curves.

Alternative I: A pure Weibull-type statistical approach is the first alternative studied. Only the random scatter of tensile strength is considered, the generic mean value of tensile strength being fixed as 3.7 MPa. The resulting size effect curve obtained by probabilistic simulation is found to have a smaller slope than the experimental data trend, in spite of the fact that an unusually low Weibull modulus ($m = 8$) is used. To explain it, note that the Weibull theory strictly applies only when the failure occurs right at the crack initiation, before any (macroscopically) significant stress redistribution with energy release takes place. This is not the case for concrete, a coarse material relative to the beam depth used. So, a nonnegligible fracture process zone must form before a macroscopic crack can form and propagate, dissipating the required fracture energy G_f per unit crack surface. Therefore, the beam, analyzed by nonlinear fracture mechanics (the crack band model, approximating the cohesive crack model) does not fail when the first element fails (as required by the weakest link model imitating the failure of a chain). Rather, it fails only after a group of elements fails, and several groups of failing elements can develop before the beam fails; see Fig. 6. The finite element simulations are able to capture this behavior thanks to the cohesive nature of softening in a crack or crack band, reflecting the energy release requirement of fracture mechanics.

Alternative II: To overcome the aforementioned problem and match the size effect data, we must take into account the randomness of fracture energy G_f. But we cannot ignore the statistical correlation of G_f to tensile strength. For lack of available data, we simply assume a very strong correlation, characterized by correlation coefficient 0.9. Such a correlation tends to cause the (macroscopic) crack propagation to begin earlier than in Alternative I. The result is shown in Fig. 4 as Alternative II. The resulting slope of the simulated size effect curve is now close to the slope of experimental data. However, the entire curve is shifted down, i.e., all the beams are weaker than they ought to be. The strong correlation between the tensile strength and fracture energy is see to cause the macroelements with a lower tensile strength to be more brittle. Therefore, the failure must localize into these macroelements.

Alternative III: In seeking a remedy, we must realize that Koide et al. measured neither the tensile strength nor the fracture energy, and that our foregoing estimates may have been too low. So, our only option is a heuristic approach. While keeping Aternative II, we are free to shift the size effect curve up by increasing the generic mean value of tensile strength and the fracture energy value. We increase them to 4 MPa and 100 N/m, respectively, and this adjustment is found to furnish satisfactory results; see Fig. 4. Although the size effect of Alternative III in the double

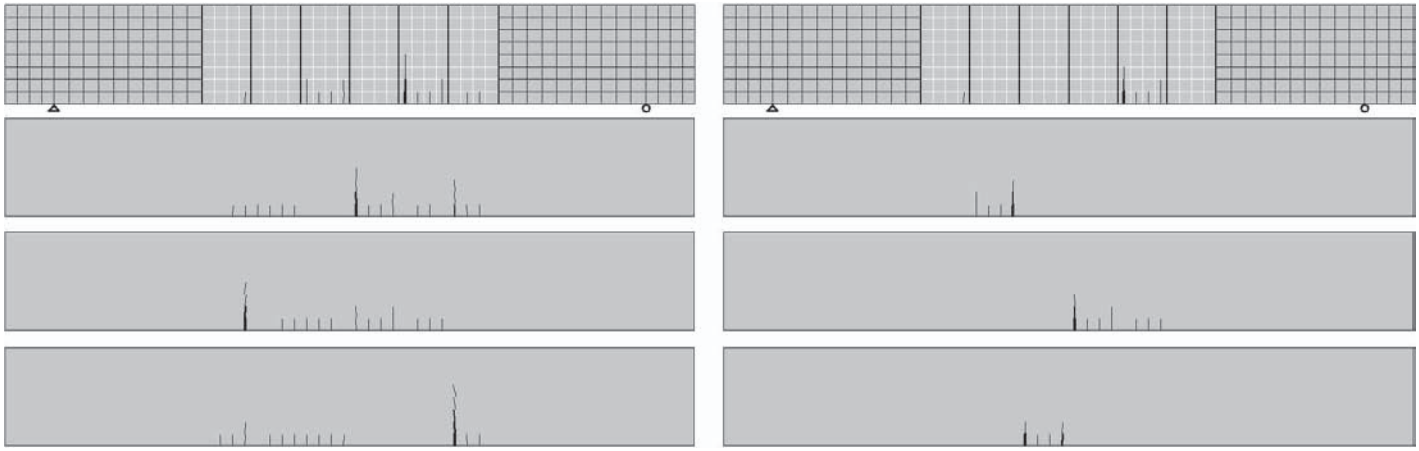

Figure 6: Macroelements and examples of random crack initiation for the first size; left: random tensile strength only, right: random and correlated tensile strength and fracture energy.

logarithmic plot is not as straight as the trend of data, the differences from the data are negligible. These small differences may have been easily caused, for example, by an insufficient size of the data set or by weak stability of load-controlled computations near the peak load, making the detection of the peak inaccurate.

In conclusion, it may be emphasized that the result of Alternative III is in excellent agreement with the previous analysis of the same data using the nonlocal Weibull theory (Bažant & Novák 2000b).

7 CONCLUDING REMARK

The factors of safety for designing against the risk of failure of concrete structures have doubtless much larger errors than the errors stemming from the inadequacies of the finite element analysis. Therefore, it would make little sense to strive for improvements of the finite element analysis with the underlying constitutive and fracture laws without at the same time addressing the limitations of the statistical theory. The present practice of finite element analysis, and even its stochastic generalization, cannot realistically cope with the statistical risk of failure because it does not provide the correct probability structure of the far-off tail of the failure probability distribution as a function of the applied load. The present analysis attempts to identify the problem and propose a way, doubtless not the only way, of achieving the correct extreme value distribution of failure probability for structures that fail in a quasibrittle manner.

APPENDIX I. INTERPLAY OF DETERMINISTIC AND STOCHASTIC LENGTH SCALES

Progress in the understanding of the uncertainties concrete failure and fracture behavior with reliability aspects has been achieved in many works; e.g. Shinozuka (1972), Mihashi & Izumi (1977), Mazars (1982), Bažant & Xi (1991), Breysse (1990), Breysse & Renaudin (1996), Carmeliet (1994), Carmeliet & Hens (1994), de Borst & Carmeliet (1996), Gutiérez (1999), Gutiérez & de Borst (1999, 2000, 2001, 2002) and others. A finite element reliability method for gradient-enhanced damage models has been formulated by Gutiérez & de Borst (1999. They considered the damage threshold as an autocorrelated random field and applied their model to quasibrittle damage in tensile double-edge-notched specimens and pull-out of steel anchors from concrete simulating damage threshold as a random field. The analysis furnished the most likely localization patterns corresponding to a chosen failure criterion and the influence of deterministic and statistical length scales on the statistical properties of structural response, size effect and damage accumulation.

The characteristic length of a nonlocal continuum governs the deterministic scaling and the autocorrelation length of random field in stochastic finite elements governs the statistical scaling. De Borst & Carmeliet (1996) showed that both the characteristic length and the correlation length are needed—the first to avoid localization, the second to characterize spatial randomness. A salient question is the relationship of these two characteristic length. Franziskonis (1998) studied this relationship analytically and Gutiérez & de Borst's (2002) numerical studies revealed very different roles of these two lengths.

APPENDIX II. STATISTICAL PROPERTIES OF NONLOCAL GENERALIZATION OF WEIBULL THEORY FOR SIZE EFFECT IN QUASIBRITTLE STRUCTURES

The statistical properties of the nonlocal generalization of Weibull theory (Eq. 7) can be derived ana-

lytically. This derivation, which is more fundamental than the direct asymptotic matching approach (Bažant & Novák 2000c), Bažant 2002) and is presented in full in Bažant (2002c), will now be briefly outlined.

Considering the nonlocal averaging domains in a nonlocal model of a structure to be analogous to the links of a chain, one may calculate the the failure probability P_f of a structure is given by $-\ln(1-P_f) = \int_V \langle \hat{\sigma}(\boldsymbol{x})/s_0 \rangle^m \, \mathrm{d}V(\boldsymbol{x})/V_r$ (Bažant & Xi 1991) where V = volume of structure, V_0 = representative volume of material, m = Weibull modulus, s_0 = scaling parameter, $\sigma(\boldsymbol{x})$ = maximum principal stress at point of coordinate vector $\boldsymbol{x}$, $\hat{\sigma}$ = nonlocal stress, V_r = representative volume of the material for which the Weibull parameters m and s_0 have been experimentally identified; and $\langle .. \rangle$ denotes the positive part of the argument. It is convenient to introduce dimensionless coordinates and variables by setting $\xi = D\xi, V_0 = l^{n_d}, V = l^{n_d} v, \mathrm{d}V(\boldsymbol{x}) = l^{n_d} \mathrm{d}v(\boldsymbol{\xi}), \sigma(\boldsymbol{x}) = \sigma_N S(\boldsymbol{\xi})$ and $l = {V_r}^{1/n_d}$ where D = size (characteristic dimension) of the structure; n_d = number of spatial dimensions in which the structure is scaled (n_d = 1, 2 or 3 for one-, two- or three-dimensional scaling); l = characteristic length of material; $\boldsymbol{\xi} = \boldsymbol{x}/D$ = dimensionless coordinate vector; and $\sigma_N = P/bD$ = nominal strength of structure (P = maximum load, b = width of structure). We consider geometrically similar structures of different sizes D, for which the corresponding points have the same dimensionless coordinate $\boldsymbol{\xi}$; then (Bažant 2002c)

$$P_f = 1 - \mathrm{e}^{-(\sigma_N/s_D)^m}, \quad {s_D}^m = \frac{{s_0}^m}{H^m} \left(\frac{l}{D} \right)^{n_d} \tag{17}$$

$$H^m = \int_v \langle \hat{S}(\boldsymbol{\xi}) \rangle^m \mathrm{d}v(\boldsymbol{\xi}) \tag{18}$$

The nonlocal stress $\hat{S}(\boldsymbol{\xi})$ used above does not permit determining the size effect analytically. Therefore, we restrict attention to large enough structures such that the nonlocal averaging domain, roughly of the same size as the fracture process zone (the zone of distributed cracking or localized damage), is small enough compared to D, though not necessarily negligible.

It can be shown in general for various definitions of the nonlocal stress and verified by the example of a three-point bend beam that the first two terms of the asymptotic expansion of the dimensionless nonlocal integral H as a power series in l/D may be written as $H = H_0(1 - c_m l/D)$, the error being of the order of $(l/D)^2$; here H_0 and c_m are dimensionless constants for geometrically similar structures. Since the term $(1 - c_m l/D)$ becomes negative for small enough l/D (which would render (17) physically meaningless), it is suitable to introduce the following equivalent approximation (which maintains the same order of asymptotic accuracy, the error being of the second order in l/D):

$$H^m = {H_0}^m (1 - c_m l/D)^m \approx {H_0}^m (1 + mrc_m l/D)^{-1/r} \tag{19}$$

(Bažant 2002c) where r is an arbitrary positive empirical constant. Eq. (17) then yields:

$$\sigma_N = [-\ln(1 - P_f)]^{1/m} s_D \quad (D \gg l) \tag{20}$$

$$s_D = \frac{s_0}{H_0} \left(\frac{l}{D} \right)^{n_d/m} \left(1 + mrc_m \frac{l}{D} \right)^{1/r} \tag{21}$$

(Bažant 2002c). This expression represents the large-size size effect law of nominal strength of structure with any specified failure probability P_f. For P_f = 0.5 it represents the large-size size effect law for the median nominal strength.

Eq. (21) gives for any D a real value of σ_N which, for any fixed P_f, decreases monotonically with D through the entire size range $D \in (-\infty, \infty)$. However, the limiting nominal strength for $D \to 0$ is infinite. From a purely empirical viewpoint, this might not be considered as objectionable because unreasonably large σ_N might result only for a hypothetical structure size D much smaller than the aggregate size. However, we prefer the small-size asymptotic properties to agree with the theoretical small-size asymptotic properties of the cohesive (or fictitious) crack model or the crack band model, or the nonlocal damage model, which imply that the value of $\hat{\sigma}_N$ for $D \to 0$ should be finite and should be approached linearly in D (Bažant 2001, 2002); this may be achieved by replacing $1/D$ with $1/(\eta l + D)$, which has no effect on the large size asymptotic properties (η = empirical coefficient of the order of 1). With this replacement, Eq. (21) becomes:

$$s_D = \frac{s_0}{H_0} \left(\frac{l}{\eta l + D} \right)^{\frac{n_d}{m}} \left(1 + mrc_m \frac{l}{\eta l + D} \right)^{\frac{1}{r}} \tag{22}$$

(Bažant 2002c) where $D \gg l$. It may be noted that the mean size effect law for mean σ_N implied by this result is not identical to (9). However, the difference is barely distinguishable in data fitting.

Similar to the classical Weibull theory, the mean and standard deviation of σ_N may be calculated as:

$$\overline{\sigma}_N = \int_0^\infty \sigma_N \mathrm{d}P_f(\sigma_N) = s_D \, \Gamma(1 + 1/m) \tag{23}$$

$$\begin{aligned} {\delta_N}^2 &= \int_0^\infty {\sigma_N}^2 \mathrm{d}P_f(\sigma_N) - {\overline{\sigma}_N}^2 \\ &= {s_D}^2 \, \Gamma(1 + 2/m) \; - \; {\overline{\sigma}_N}^2 \end{aligned} \tag{24}$$

Accordingly, the coefficient of variation of σ_N (for $D \gg l$)

$$\omega_N = \frac{\delta_N}{\overline{\sigma}_N} = \sqrt{\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1} \qquad (25)$$

is asymptotically independent of structure size D, and is given by the same expression as in Weibull theory. But this is true only for large enough D/l.

While Eq. (23) with (22) gives a realistic size effect formula for the mean strength throughout the full size range $D \in (0, \infty)$, the coefficient of variation in (25) and the entire probability distribution P_f given by (17) or (20) with (22) are certainly invalid for small enough D/l. The reason is that the failure of small structures with a cohesive crack or crack band approaches, for $D \to 0$, the case of elastic body with a plastic crack for which the failure probability has an entirely different structures. Since the failure of such a small structure must be non-propagating (i.e., simultaneous along the entire failure surface), the failure probability should obey Daniels' 'fiber-bundle' (parallel coupling) model rather than extreme value statistics. So, for $D \to 0$, the size effect on the mean nominal strength must asymptotically vanish. The failure probability distribution according to Daniels' model is (in agreement with the central limit theorem of probability) the gaussian distribution, except for the far-out tails (this may explain why the old studies based on measuring ω_N gave for concrete $m \approx 12$ instead of the correct value $m \approx 24$ based on size effect measurements). Denoting $\sigma_N \propto F_g(P_f)$ = inverse of cumulative gaussian (normal) distribution for the small size limit, one may conjecture roughly the following general size effect law giving the nominal strength of any specified failure probability P_f for the full size range (Bažant 2002c):

$$\sigma_N = s_D \left\{ \frac{l^u}{l^u + D^u} F_g(P_f) \right. \qquad (26)$$

$$\left. + \frac{D^u}{l^u + D^u} \left[-\ln(1 - P_f) \right]^{1/m} \right\}$$

where u is a certain suitable positive exponent. According to Daniels' (1945) model (for infinitely many fibers in the bundle), the gaussian cumulative distribution to which $F_g(P_f)$ is inverse should be such that its mean σ_N is independent of D but its coefficient of variation is proportional to $1/\sqrt{D}$.

Acknowledgment: Partial financial support under Grant ONR-N00014-02-I-0622 from Office of Naval Research (monitored by Dr. Yapa D.S. Rajapakse) is gratefully acknowledged. The second author gratefully acknowledge the support from the Grant Agency of the Czech Republic under the grant No. 103/03/1350. Authors thanks to M. Vořechovský for numerical simulation of four-point-bend flexural tests.

REFERENCES

Bažant, Z.P. 1976. Instability, ductility, and size effect in strain-softening concrete. *J. Engng. Mech. Div., Am. Soc. Civil Engrs.*, 102 (EM2), 331–344; disc. 103, 357–358, 775–777, 104, 501–502.

Bažant, Z.P. 1984. Size effect in blunt fracture: Concrete, rock, metal. *J. of Engng. Mechanics*, ASCE, 110, 518–535."

Bažant, Z.P. 1986. Mechanics of distributed cracking. *Appl. Mech. Reviews ASME*, 39, 675–705.

Bažant, Z.P. 1987. Fracture energy of heterogeneous material and similitude. Preprints, SEM-RILEM *Int. Conf. on Fracture of Concrete and Rock* (held in Houston, Texas, June 1987), ed. by S.P. Shah and S.E. Swartz, publ. by SEM (Soc. for Exper. Mech.) 390–402.

Bažant, Z.P., Editor 1992. *Fracture Mechanics of Concrete Structures*, Proc., First Intern. Conf. (FraMCoS-1), held in Breckenridge, Colorado, June 1–5, Elsevier, London (1040 pp.).

Bažant, Z.P. 1993. Scaling Laws in Mechanics of Failure. *J. of Engrg. Mech., ASCE*, 119 (9), 1828–1844.

Bažant, Z.P. 1997. Scaling of quasibrittle fracture: Asymptotic analysis. *Int. J. of Fracture* 83 (1), 19–40.

Bažant, Z.P. 1998. Size effect in tensile and compression fracture of concrete structures: computational modeling and design. *Fracture Mechanics of Concrete Structures* (3rd Int. Conf., FraMCoS-3, held in Gifu, Japan), H. Mihashi and K. Rokugo, eds., Aedificatio Publishers, Freiburg, Germany, 1905–1922.

Bažant, Z.P. 1999. Size effect on structural strength: a review. *Archives of Applied Mechanics* (Ingenieur-Archiv, Springer Verlag) 69, 703–725 (75th Anniversary Issue).

Bažant, Z.P. 2001a. Size effects in quasibrittle fracture: Apercu of recent results. *Fracture Mechanics of Concrete Structures* (Proc., FraMCoS-4 Int. Conf., Paris), R. de Borst et al., eds., A.A. Balkema Publishers, Lisse, Netherlands, 651–658.

Bažant, Z.P. 2001b. Probabilistic modeling of quasibrittle fracture and size effect. (principal plenary lecture, Proc., 8th Int. Conf. on Structural Safety and Reliability (ICOSSAR), held at Newport Beach, Cal., 2001), R.B. Corotis, ed., Swets & Zeitinger (Balkema), 1–23.

Bažant, Z.P. 2002c. Statistical properties of nonlocal generalization of Weibull theory for size effect in quasibrittle structures. Preliminary Report to ONR No. 02-12-C699s, Dept. of Civil Engrg., Northwestern University, Evanston, Dec. (also submitted to JEM-ASCE, and in condensed form to Proc., ICASP, Berkeley, July 2003.

Bažant, Z.P. & Chen, E.-P. 1997. Scaling of structural failure. *Applied Mechanics Reviews ASME* 50 (10), 593–627.

Bažant, Z.P. & Frangopol, D.M. 2002. Size effect hidden in excessive dead load factor. *J. of Structural Engrg. ASCE* 128 (1), 80–86.

Bažant, Z.P. & Kazemi, M.T. 1990. Size effect in fracture of ceramics and its use to determine fracture energy and effective process zone length. *J. of American Ceramic Society* 73 (7), 1841–1853.

Bažant, Z.P. & Novák, D. 2000a,b. Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. I. Theory, and II. Application. *J. of Engrg. Mech.* ASCE 126 (2), 166–174 and 175–185.

Bažant, Z.P. & Novák, D. 2000c. Energetic-statistical size effect in quasibrittle failure at crack initiation. *ACI Materials Journal* 97 (3), 381–392.

Bažant, Z.P. & Novák, D. 2001a. Proposal for standard test of modulus of rupture of concrete with its size dependence. *ACI Materials Journal* 98 (1), 79–87.

Bažant, Z.P. & Novák, D. 2001b. Nonlocal Weibull theory and size effect in failures at fracture initiation. *Fracture Mechanics of Concrete Structures* (Proc., FraMCoS-4 Int. Conf., Paris), R. de Borst et al., eds., Swets & Zeitlinger (A.A. Balkema Publishers), Lisse, Netherlands, 659–664.

Bažant, Z.P. & Planas, J. 1998. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials*. CRC Press, Boca Raton, Florida.

Beremin, F.M. 1983. A local criterion for cleavage fracture of a nuclear pressure vessel steel. *Metallurgy Transactions A*, 14, 2277–2287.

Breysse, D. 1990. A probabilistic formulation of the damage evaluation law. *Struct. Safety*, 8, 311–325.

Breysse, D. & Renaudin, P. 1996. On the influence of local disorder on size effect. *Size-Scale Effects in the Failure Mechanisms of Materials and Structures* (Proc., IUTAM Symp. held in 1994 at Politecnico di Torino), ed. A. Carpinteri, E & FN Spon, London, 187–199.

Carmeliet, J. 1994. On stochastic descriptions for damage evolution in quasi-brittle materials. *DIANA Computational Mechanics*, G.M.A. Kusters and M.A.N. Hendriks (eds.)

Carmeliet, J. & Hens, H. 1994. Probabilistic nonlocal damage model for continua with random field properties. *ASCE J. of Engrg. Mech.*, 120(10), 2013–2027.

Červenka, V. & Pukl, R. 2002. ATENA - Computer Program for Nonlinear Finite Element Analysis of Reinforced Concrete Structures", Červenka Consulting, Prague.

de Borst, R. & Carmeliet, J. 1996. Stochastic approaches for damage evoluation in local and non-local continua. In *Size-Scale Effects in the Failure Mechanism of Materials and Structures*, (ed. A. Carpinteri), E & FN Spon, London, 242–257.

Deodatis, G. & Shinozuka, M. 1991. Weighted integral method. II. Response variability and reliability. *J. of Engrg. Mech. ASCE* 117 (8), 1865–1877.

Engelund, S. & Rackwitz, R. 1992. On predictive distribution functions for the three asymptotic extreme value distributions. *Structural Safety* 11, 255–258.

Epstein, B. 1948. Statistical aspects of fracture problems. *J. of Applied Physics* 19, 140–147.

Fisher, R.A. & Tippett, L.H.C. 1928. Limiting forms of the frequency distribution of the largest and smallest member of a sample. *Proc., Cambridge Philosophical Society* 24, 180–190.

Frantzisconis, G.N. 1998. Stochastic modelling of heterogeneous materials - A process for the analysis and evaluation of alternative formulations. *Mechanics of Materials.*, 27, 165–175.

Fréchet, M. 1927. Sur la loi de probabilité de l' écart maximum. *Ann. Soc. Polon. Math.* 6, 93.

Freudenthal, A.M. 1956. Physical and statistical aspects of fatigue. in *Advance in Applied Mechanics*, Vol. 4, Academic Press, 117–157.

Freudenthal, A.M. 1968. Statistical approach to brittle fracture. Chapter 6 in *Fracture*, Vol. 2, ed. by H. Liebowitz, Academic Press, 591–619.

Freudenthal, A.M., and Gumbel, E.J. 1953. On the statistical interpretation of fatigue tests. *Proc. of the Royal Society (London)* A, Vol. 216, 309-332.

Freudenthal, A.M. & Gumbel, E.J. 1956. Physical and statistical aspects of fatigue. in *Advances in Applied Mechanics*, Vol. 4, Academic Press, 117–157.

Freudenthal, A.M. 1981. *Selected Papers*, Am. Soc. of Civil Engrs., New York.

Ghanem, R.G. & Spanos, P.D. 1991. *Stochastic finite elements: a spectral approach.* Springer-Verlag, New York.

Gnedenko, B.V. 1943. Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* 44, p. 423.

Gnedenko, B.V. & Kolmogorov, A.N. 1954. *Limit ditributions of sums of random variables*. Addison Wesley, Cambridge, Massachussetts.

Gumbel, E.J. 1954. *Statistical Theory of Extreme Values and Some Practical Applications*, Nat. Bureau of Standards, Washington, D.C., Appl. Math. Series No. 33

Gumbel, E.J. 1958. *Statistics of Extremes*. Columbia University Press, New York.

Gutiérez, M.A. 1999. *Objective simulation of failure in heterogeneous softening solids.* Dissertation, Delft University of Technology.

Gutiérez, M. A. & de Borst, R. 1999. Deterministic and stochastic analysis of size effects and damage evaluation in quasi-brittle materials. *Archive of Applied Mechanics* 69, 655-676.

Gutiérez, M.A. & de Borst, R. 2000. Stochastic aspects of localized failure: material and boundary imperfections. *Int. J. of Solids and Structures* 37, 7145–7159.

Gutiérez, M.A. & de Borst, R. 2001. Deterministic and probabilistic material length scales and their role is size-effect phenomena. *Report*, Koiter Institute, Delft University of Technology, Delft, Netherlands; submitted for publication.

Gutiérez, M.A. & de Borst, R. 2002. Deterministic and probabilistic material length-scales and their role in size-effect phenomena. *ICOSSAR01, CD-ROM Proc. of Int. Conf. on Structural Safety and Reliability*, Newport Beach, California, Balkema.

Hillerborg, A., Modéer, M. & Petersson, P.E. 1976. Analysis of crack formation and crack growth in concrete by

means of fracture mechanics and finite elements. *Cem. Concr. Res.* 6, 773–782.

Hinch (1991). *Perturbation Methods*. Cambridge University Press, Cambridge, UK.

Iguro, M., Shiyoa, T., Nojiri, Y. & Akiyama, H. 1985. ”Experimental studies on shear strength of large reinforced concrete beams under uniformly distributed load, *Concrete Library International, Japan Soc. of Civil Engrs.* No. 5, 137–154 (translation of 1984 article in Proc. JSCE).

Kleiber, M. & Hien, T.D. 1992. The stochastic finite element method. J. Wiley, New York.

Koide, H., Akita, H. & Tomon, M. 1998. Size Effect on Flexural Resistance due to Bending Span of Concrete Beams. Fracture Mechanics of Concrete Structures (Proc., 3rd Int. Conf., FraMCoS-3, held in Gifu, Japan), H. Mihashi and K. Rokugo, eds., Aedificatio Publishers, Freiburg, Germany, 2121-2130.

Koide, H., Akita, H. & Tomon, M. 2000. Probability Model of Flexural Resistance on Different Lengths of Concrete Beams. Application of Statistics and Probability (Proc., 8th Int. Conf., ICASP-8, held in Sydney, Australia 1999), R. E. Melchers and M. G. Stewart, eds., Balkema, Rotterdam, Vol. 2, 1053-1057.

Lei, Y., O'Dowd, N.P., Busso, E.P. & Webster, G.A. 1998. Weibull stress solutions for 2-D cracks in elastic and elastic-plastic materials. *Int. J. of Fracture* 89, 245-268.

Liu, W.K., Mani, A. & Belytschko, T. 1987. Finite element methods in probabilistic mechanics. *Probabilistic Engineering Mechanics* 2 (4), 201–213.

von Mises, R. 1936. La distribution de la plus grande de n valeurs. *Rev. Math. Union Interbalcanique* 1, p. 1.

Novák, D. et al. 2002. FREET - Feasible Reliability Engineering Efficient Tool. Report to Červenka Consulting, Prague, Brno University of Technology, Faculty of Civil Engineering, Institute of Structural Mechanics.

Novák, D., Rusina, R. & Vořechovský 2003. Small-sample statistical analysis - software FREET. *Proc., ICASP 9*, San Francisco, USA, in press.

Novák, D., Vořechovský, M., Pukl, R. & Červenka, V. 2001. Statistical nonlinear analysis - size effect of concrete beams. *Fracture Mechanics of Concrete and Concrete Structures* (Proc., 4th Int. Conference, FraMCoS-4), Cachan, France, Balkema, Lisse, Netherlands.

Peerlings, R.H.J., de Borst, R., Brekelmans, W. & Vre, J. 1996. Gradient enhanced damage for quasibrittle materials. *Int. J. of Num. Meth. in Engrg.* 39, 3391–3403.

Peirce, F.T. 1926. Tensile tests of cotton yarns—V. The weakest link. *J. Textile Inst.* 17, 355–368.

Pukl R., Strauss A., Novák, D.,Červenka V. & Bergmeister K. 2003. Probabilistic-based assessment of concrete structures using non-linear fracture mechanics. *Proc., ICASP 9*, San Francisco, USA, in press.

Ruggieri, C. & Dodds, R.H. 1996. Transferability model for brittle fracture including constraint and ductile tearing effects—a probabilistic approach. *Int. J. of Fracture* 79, 309–340.

Saibel, E. 1969. Size effect in structural safety. *Proc., Int. Conf. on Structure, Solid Mechanics and Engineering Design in Civil Engineering Materials*, Southampton, Part I, 125–130.

Schuëller, G. I. 1997a,b. Structural reliability — recent advances. (Freudenthal lecture), *Proc., ICOSSAR-97, Kyoto, Japan*, Balkema, Rotterdam, Vol. I, 3–35.

Schuëller, G. I. 1997. A state-of-art report on computational stochastic mechanics. *Probabilistic Engineering Mechanics* 12 (4), 199-321.

Shinozuka, M. & Deodatis, G. 1988. Response variability of stochastic finite elements. *J. of Engrg. Mech, ASCE* 118 (3), 499–519.

Shioya, Y. & Akiyama, H. 1994. Application to design of size effect in reinforced concrete structures. in *Size Effect in Concrete Structures* (Proc. of Intern. Workshop in Sendai, 1993), ed. by H. Mihashi, H. Okamura and Z.P. Bažant, E & FN Spon, London, 409–416.

Takada, T. 1990. Weighted integral method in multidimensional stochastic finite element analysis. *Prob. Engrg. Mech.* 5 (4), 158–166.

Tippett, L.H.C. 1925. On the extreme individuals and the range of samples. *Biometrika* 17, 364.

Vořechovský & Novák, D. 2003. Small-sample statistical analysis – software FREET. *Proc., ICASP 9*, San Francisco, USA, in press.

Walraven, J. 1995. Size effects: their nature and their recognition in building codes. *Studi e Ricerche* (Politecnico di Milano) 16, 113-134.

Walraven, J. & Lehwalter 1994. Size effects in short beams loaded in shear. *ACI Structural Journal* 91 (5), 585–593.

Weibull, W. 1939. The phenomenon of rupture in solids. Proc., Royal Swedish Institute of Engineering Research (Ingenioersvetenskaps Akad. Handl.) 153, Stockholm, 1–55.