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Mechanics Based Statistical Prediction of Structure Size and Geometry Effects on Safety Factors for Composites and Other Quasibrittle Materials

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For a rational determination of safety factors, it is necessary to establish the probability density distribution function (pdf) of the structural strength. For perfectly ductile and perfectly brittle materials, the proper pdf's of the nominal strength of structure are known to be Gaussian and Weibullian, respectively, and are invariable with structure size and geometry. However, for quasibrittle materials, many of which came recently to the forefront of attention, the pdf has recently been shown to depend on structure size and geometry, varying gradually from Gaussian pdf with a remote Weibull tail at small sizes to a fully Weibull pdf at large sizes. This recent result is reviewed, and then mathematically extended in two ways: 1) to a mathematical description of structural lifetime as a function of applied (time-invariable) nominal stress, and 2) to a mathamatical description of the statistical parameters of the pdf of structural strength as a function of structure size and shape. Experimental verification and calibration is relegated to a subsequent journal article.

I. Introduction

The design of engineering structures such as aircraft, bridges, dams, nuclear containments, and ships must ensure an extremely low failure probability[1–3] (such as $P_f = 10^{-6}$

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to 10^{-7}). The same is required for micro-electronics and bio-medical devices, as well as for the lifetime of structures. In this range of P_f, it is virtually impossible to determine the tail of the probability distribution function (pdf) of load F by histogram testing. It is inevitable to rely on a theory to be verified indirectly. Its formulation has been a fundamental problem of failure mechanics, in which only two limiting failure types are now adequately understood: • 1) perfectly ductile (plastic) failures, for which (because of the central limit theorem of probability theory) the pdf of F is necessarily Gaussian, or normal (except in far-out tails) since F is essentially a weighted sum of the strength contributions from all the representative volume elements (RVE) of the material lying on the failure surface; and • 2) perfectly brittle failures, which are decided by the failure at one material point and thus follow the weakest-link model which gives Weibull distribution. In these limit cases, which include fine-grained ceramics and fatigue-embrittled metallic structures, the required pdf tail is estimated with high confidence and is independent of structure size and geometry.

Recently[4–7] it has been shown how the problem, including its scaling aspect, can be solved for the broad and increasingly important class of quasibrittle structures,[8–10] whose failure behavior lies between these two extremes. Although the matrix of a quasibrittle material is brittle, its heterogeneous microstructure causes the RVE and the fracture process zone (FPZ) not to be negligibly small compared to the characteristic size D (or cross-section dimension) of the structure. This includes materials such concrete (the archetypical, by now classical, case), rock, stiff soils or snow, sea ice, wood, paper and carton, as well as modern 'high-tech' materials such as toughened ceramics, fiber composites and rigid foams, or biological materials such as bone, cartilage, dentin and sea shells. Since every brittle structure becomes quasibrittle when scaled down to $D <$ circa $1000\ l_0$, where $l_0 =$ RVE size, the problem becomes important for nano- and micro-meter scale devices (nano-composites, MEMS, thin films).

Attention is here focussed on structures of positive geometry[8]—a typical and dangerous case in which the removal of one RVE suffices to cause failure (under constant load). When $D/l_0 \to \infty$, the geometry is positive if the derivative of the stress intensity factor with respect to crack length a at constant F is positive. D. According to the classical statistical theory of brittle failure,[11] a structure of positive geometry fails as soon as the random material strength is reached at one point of the structure. The quasibrittle structures, in which the RVE size is not negligible, fail when the strength of one RVE as a whole is exhausted. Hence, the number of RVEs in the structure is finite, and one must use the weakest-link model with a *finite* number, N, of links in the chain. The RVE must here be defined not by homogenization but as the smallest element whose failure will cause the whole structure to fail.[7] Typically, the RVE size, l_0, is about the double or triple of the maximum inhomogeneity size (or grain size).

For the background literature, it is appropriate to cite, at least, Ref. 1–3, 11–41. For a detailed discussion of relevant previous works, see Ref. 7.

II. Conclusions from Previous Work

In two previous studies,[6,7] the following conclusions were reached:

1. The understrength part of safety factors for quasibrittle structures cannot be constant, as generally assumed in practice, but must be varied with the size as well geometry of the structure geometry.

2. The tail of the cumulative density function (cdf) of strength of RVE of any material (whether brittle or plastic) must be a power law. The physical reason is that the failure of interatomic bonds is a thermally activated process governed by transition state theory and with stress-dependent activation energy barriers. Furthermore, this property, rather than the statistics of material flaws, provides a sufficient physical justification for Weibull cdf.

3. The threshold of power-law tail and of the Weibull cdf of strength must be zero because, according to Maxwell-Boltzmann distribution of atomic thermal energies, the threshold stress for the net rate of interatomic bond breaks is zero (Fig. 1a).

4. The physical meaning of Weibull modulus m is the number of dominant bonds that must be severed, or the number of matrix connections between adjacent major inhomogeneities that must fail, in order to cause failure of the RVE. This number must in some way depend on the spatial packing of inhomogeneities in the RVE.

5. In any statistical model consisting of series and parallel couplings (Fig. 1b) the power-law tail of cdf is preserved, beginning with the power law of exponent 1 on the atomic scale. While the series coupling preserves the exponent value, in parallel coupling the exponents are additive and can thus be raised to high values. This is the reason why the Weibull modulus m is so high, ranging from 10 to 50.

6. The multiplier (or amplitude) of the power-law tail of the cdf of strength of quasibrittle structures is the same function of absolute temperature T, load duration τ and activation energy Q as the mutliplier indicated by Maxwell-Boltzmann distribution for the rate of interatomic bond breaks (Fig. 1a).

7. The statistical model for RVE can include parallel connections of no more than 2 elements on scales close to macroscales (with power-law tail exponent greater than about 6), and 3 elements on lower scales (with a smaller power-law tail exponent), or else the power-law tail of cdf of RVE strength would be so remote that the Weibull distribution would never be observed in practice.

8. While the power-law tail exponent of a chain is equal to the lowest exponent among its links, the power-law tail exponent of a bundle (Fig. 1b) is equal to the sum of the power-

law tail exponents of all the parallel fibers in a bundle, regardless of whether they are brittle, plastic or softening. While the probability range of power-law tail increases with the length of a chain, it drastically decreases with the number of fibers coupled in parallel.

9. A sufficiently long power-law tail of RVE strength can be reconciled with Maxwell-Boltzmann distribution only if the RVE is statistically modelled by a hierarchy of parallel and series couplings, consisting of bundles of sub-chains of sub-bundles of sub-sub-chains of sub-sub-bundles, etc., down to the atomic scale. The Weibull modulus is equal to the minimum number of cuts of elementary bonds needed to separate the hierarchical model into two parts (Fig. 1b). The cdf of RVE strength cannot be modelled by a bundle with a finite number of elements following the Maxwell-Boltzmann distribution, and quasibrittle structures cannot be modelled as a chain of bundles. Otherwise the power-law tail of RVE would be far too remote for ever generating Weibull cdf for the strength of real structures.

10. For the sake of engineering computations, the cdf of random strength of a RVE may be considered to have a Weibull left tail grafted onto a Gaussian core at the failure probability of about 0.001 (or between 0.0001 and 0.01). With increasing structure size, the grafting point moves to higher failure probabilities as a function of the equivalent number of RVEs, in a way than can be described by treating the structure as a chain of finite RVEs. Although the Gaussian and Weibull cdf hardly differ in looking at experimental histograms, the point of $P_f = 10^{-6}$ is for the latter, at the same coefficient of variation, almost twice as far from the mean than it is for the former (Fig. 1c).

11. For the mean response of not too small structures, the chain-of-RVEs model gives similar results as the previously developed nonlocal Weibull theory.[19] The mean behavior is, on not too small scales, essentially equivalent to that of the cohesive crack model, crack band model and nonlocal damage models.

12. The reason that a nonzero threshold was found preferable in previous studies of coarse-grained ceramics and concrete can be traced to the fact that the strength histograms of these materials exhibit a kink separating a lower Weibull segment from an upper Gaussian segment. Assuming a finite threshold improves the fit of these histograms but the upper segment still cannot be fitted closely. The chain-of-RVEs model removes this problem. Its prediction fits both segments of the experimental histograms very well (Fig. 1d,e).

13. Two ways of experimental calibration and verification are possible: 1) Fit the mean size effect curve, particularly its deviation from the Weibull size effect for small sizes (Fig. 1f). 2) Fit the strength histograms with kinks for at least two significantly different sizes, and possibly different shapes (Fig. 1e). Each way suffices to determine all the parameters.

The objective of this paper is to derive analytical expressions for the size and shape dependence of the mean and variance of structure strength, and to extend the theory to structural lifetime.

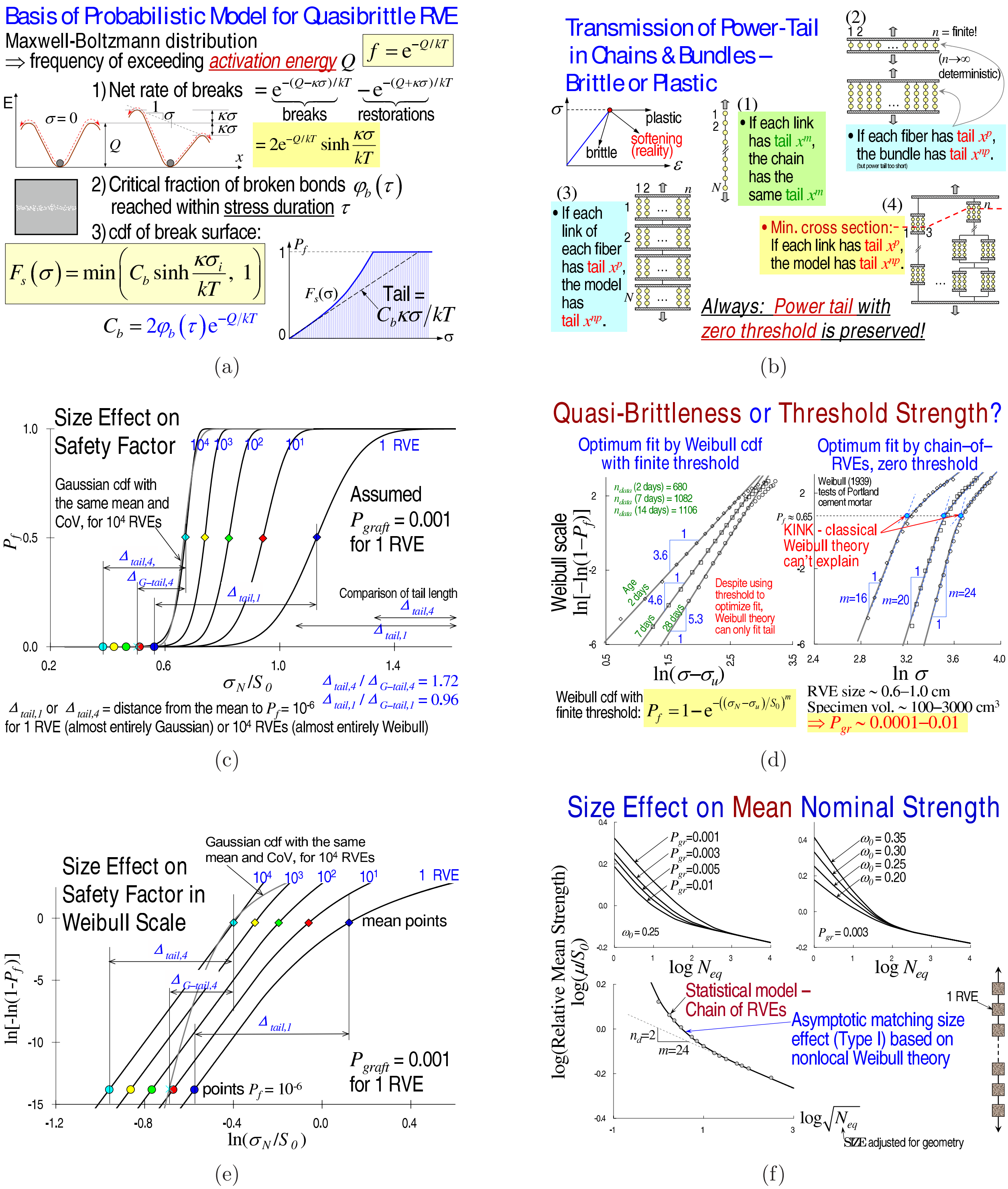

Figure 1. Models and Distributions. Top left: Activation energy barrier and corresponding cdf tail. Top right: cdf of chain-of-RVEs. Middle left: Variation of cdf with increasing size. Middle right: Fit in Weibull scale of Weibull's experimental histograms of mortar strength by Weibull cdf with a finite threshold σ_u and by chain-of-RVEs model. Bottom left: Distances from mean to points $P_f = 10^{-6}$ for various sizes. Bottom right: Size effect of chain-of-RVE model, deviating from power law.

III. Review of Size Effect in Weakest Link Model and Its Asymptotics

We will consider geometrically similar structures of different sizes D, representing the characteristic dimension of the structure. We will restrict consideration to structures of positive geometry. This is a broad class of structures, for which the derivative of the energy release rate with respect to the crack length at constant load P is positive. These are structures that fail (under load control) as soon as the full fracture process zone (FPZ) forms and a distinct continuous macro-crack begins to grow. Let $\sigma_{\mathsf{N}} = P/bD$ = nominal stress in a structure, where P = applied load (or parameter of the load system) and b = structure width. For geometrically similar elastic or elasto-plastic structures, σ_{N} at maximum load is independent of structure size, and therefore a decrease of τ with structure size is called the size effect.

From the viewpoint of failure statistics, a structure of positive geometry may be modeled as a chain (1 in Fig. 1b or bottom right in Fig. 1f), which is known as the weakest link model (the positive geometry means that the partial derivative of energy release rate with respect to crack length is positive,[8] and in the case the structure fails as soon as the FPZ, roughly equal to one RVE, is fully formed). For such structures, the representative volume element (RVE) must be defined as the smallest material volume whose failure causes the whole structure to fail.[6,7] The size of the RVE can be considered equal to the width of the FPZ and typically equals 2 to 3 material inhomogeneity sizes. If one RVE fails, the whole structure fails, i.e., the strength of the chaink is decided by its weakest element, or link, which is called the weakest link model. In our interpretation, each link corresponds to one RVE, and so we have a chain-of-RVE model (bottom right in Fig. 1f).

It has been shown[6,7] that the strength one RVE of a quasibrittle material must have a composite cumulative distribution function (cdf) having a broad Gaussian core onto which a Weibull tail is grafted at the probability of about $P_g \approx 0.001$ (grafting probability). Since, in the weakest-link model, $\sigma_{\mathsf{N}} = \sigma$ = stress in each link (i.e., one RVE), the Gaussian core may be written as

$$P_1 = \Phi(x), \qquad x = \frac{\sigma_{\mathsf{N}} - \mu}{s_0} \tag{1}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-x^2/2} \mathrm{d}x \tag{2}$$

where μ = mean, s_0 = standard deviation, and $\Phi(x)$ is the error function representing the standard (unit) Gaussian (or normal) cumulative distribution function (cdf). Because P_g is, for a single RVE, very small, the Weibull tail is nearly identical to a power function, i.e.,

$$P_f = (\sigma_{\mathsf{N}}/s_W)^m \tag{3}$$

where s_W = scaling parameter and m = material constant or Weibull modulus (its value is roughly equal to the number of dominant micro-cracks in the RVE that must fail to cause the whole RVE to fail[6,7]).

In a chain of N RVEs, simulating the failure of a large structure, the whole chain survives if all the RVEs survive. So, according to the joint probability theorem, $1 - P_N = (1 - P_1)^N$ where P_N is the failure probability of the whole chain. Hence

$$P_N(\sigma_{\mathsf{N}}) = 1 - [1 - P_1(\sigma_{\mathsf{N}})]^N \tag{4}$$

Here N should be interpreted not as the actual number of RVEs in the structure, $N_{\mathsf{actual}} = V/{l_0}^{n_d}$, but as the equivalent number of RVEs in the structure, which is obtained as[7]

$$N = V/({l_0}^{n_d}\Psi), \qquad \Psi = \int_V \langle\tilde{\sigma}(\boldsymbol{\xi})\rangle^m \mathrm{d}V(\boldsymbol{\xi}) \tag{5}$$

Here V = volume of the structure, $\langle x\rangle = \max(0, x)$, l_0 = RVE size = material characteristic length, n_d = number of dimensions in which the failure is scaled (1, 2, or 3), $\boldsymbol{\xi}$ = dimensionless coordinate vector (independent of structure scaling), and Ψ = geometry factor, causing that the RVEs receiving small stress contribute to the equivalent number N very little.

An exact analytical solution of cdf of the strength of a structure of any size seems impossible, but we will show that approximate analytical formulas for the mean and coefficient of variation can be obtained by asymptotic matching.

IV. Small-Size Asymptotics of Mean Strength

First consider the small-size asymptotics, for $N \to 1$ (it is convenient to treat N as a continuous variable). Then we may write:

$$1 - P_N(\sigma_{\mathsf{N}}) = [1 - P_1(\sigma_{\mathsf{N}})][1 - P_1(\sigma_{\mathsf{N}})]^{N-1} = [1 - P_1(\sigma_{\mathsf{N}})]\mathrm{e}^{(N-1)\ln[1-P_1(\sigma_{\mathsf{N}})]} \tag{6}$$

Noting that $\mathrm{e}^x \approx 1 + x$ for $x \ll 1$, we have the small-size approximation:

$$1 - P_N(\sigma_{\mathsf{N}}) \approx [1 - P_1(\sigma_{\mathsf{N}})]\{1 + (N-1)\ln[1 - P_1(\sigma_{\mathsf{N}})]\} \tag{7}$$

Because, for $N \approx 1$, almost the entire strength distribution is Gaussian,

$$1 - P_N(\sigma_\mathsf{N}) \approx \Psi(\sigma_\mathsf{N})\,[1 + (N-1)\ln\Psi(\sigma_\mathsf{N})] \tag{8}$$

$$\text{where} \qquad \Psi(\sigma_\mathsf{N}) = 1 - \Phi(x) \tag{9}$$

This approximation should be sufficient for determining the mean nominal strength $\bar{\sigma}_\mathsf{N}$ for $N \to 1$. Since strength σ cannot be negative,

$$\sigma = \int_0^\infty \sigma p_f(\sigma)\mathrm{d}\sigma = \int_0^\infty \sigma\frac{\mathrm{d}P_f}{\mathrm{d}\sigma}\mathrm{d}\sigma = \int_0^1 \sigma\mathrm{d}P_f(\sigma) = \int_0^\infty [1 - P_N(\sigma)]\mathrm{d}\sigma \tag{10}$$

where $p_f(\sigma)$ = probability density function (pdf) of strength. So, Eq. (8) yields

$$\text{for } N \text{ close to 1:} \qquad \bar{\sigma}_\mathsf{N} = A - (N-1)B \tag{11}$$

$$\text{where} \qquad A = \int_0^\infty \Psi(\sigma_\mathsf{N})\mathrm{d}\sigma_\mathsf{N} \tag{12}$$

$$B = -\int_0^\infty \Psi(\sigma_\mathsf{N})\ln\Psi(\sigma_\mathsf{N})\mathrm{d}\sigma_\mathsf{N} \tag{13}$$

V. Small-Size Asymptotics of Coefficient of Variation

The coefficient of variation ω of the strength distribution for N close to 1 may be determined as follows:

$$\begin{aligned}
\omega^2 &= \frac{1}{\bar{\sigma}_\mathsf{N}^2}\int_0^1 {\sigma_\mathsf{N}}^2\mathrm{d}P_N(\sigma_\mathsf{N}) - 1 \\
&\approx \frac{1}{A^2}\left[1 - (N-1)\frac{B}{A}\right]^{-2}\int_0^\infty \sigma_\mathsf{N}^2\frac{\mathrm{d}P_N}{\mathrm{d}\sigma_\mathsf{N}}\,\mathrm{d}\sigma_\mathsf{N} \\
&\approx \frac{1}{A^2}\left[1 - (N-1)\frac{B}{A}\right]^{-2}\int_0^\infty \sigma_\mathsf{N}^2\frac{\mathrm{d}}{\mathrm{d}\sigma_\mathsf{N}}\Big(1\ - [1 - P_1(\sigma_\mathsf{N})]\{1 \\
&\quad + (N-1)\ln[1 - P_1(\sigma_\mathsf{N})]\}\Big)\mathrm{d}\sigma_\mathsf{N} - 1 \\
&\approx \frac{1}{A^2}\left[1 + 2(N-1)\frac{B}{A}\right]\int_0^\infty \sigma_\mathsf{N}^2\Big(\frac{\mathrm{d}P_1(\sigma_\mathsf{N})}{\mathrm{d}\sigma_\mathsf{N}}\{1 + (N-1)\ln[1 - P_1(\sigma_\mathsf{N})]\} \\
&\quad + (N-1)\frac{\mathrm{d}P_1(\sigma_\mathsf{N})}{\mathrm{d}\sigma_\mathsf{N}}\Big)\mathrm{d}\sigma_\mathsf{N} - 1
\end{aligned} \tag{14}$$

Denoting

$$\phi(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi} \tag{15}$$

which is the standard Gaussian (or normal) pdf, we have

$$\mathrm{d}P_1(\sigma_\mathsf{N})/\mathrm{d}\sigma_\mathsf{N} \approx \mathrm{d}\Phi(x)/\mathrm{d}x = \phi(x)/s_0, \qquad x = (\sigma_\mathsf{N} - \mu)/s_0 \tag{16}$$

$$\omega^2 \approx G - (N-1)H \quad \text{or} \quad \omega \approx \sqrt{G}[1-(N-1)H/2G] \tag{17}$$

in which

$$G = \frac{1}{A^2 s_0}\int_0^\infty \sigma_{\mathsf{N}}^2 \phi(x) \mathrm{d}\sigma_{\mathsf{N}} - 1, \qquad x = \frac{\sigma_{\mathsf{N}} - \mu}{s_0} \tag{18}$$

$$\begin{aligned} H = & -\frac{2B}{A^3 s_0}\int_0^\infty \sigma_{\mathsf{N}}^2 \phi(x)\Big\{N + (N-1)\ln[1 - P_1(\sigma_{\mathsf{N}})]\Big\}\mathrm{d}\sigma_{\mathsf{N}} \\ & -\frac{1}{A^2 s_0}\int_0^\infty \sigma_{\mathsf{N}}^2 \phi(x)\Big\{1 + \ln[1 - P_1(\sigma_{\mathsf{N}})]\Big\}\mathrm{d}\sigma_{\mathsf{N}} \end{aligned} \tag{19}$$

VI. Large-Size Asymptotics of Mean and Coefficient of Variation

Second, consider the asymptotic variation of $\bar{\sigma}_{\mathsf{N}}$ and ω for $N \to \infty$. In this case $P_1(\sigma_{\mathsf{N}})$ converges to the Weibull distribution, i.e.,

$$P_N(\sigma_{\mathsf{N}}) = 1 - \left[1 - \frac{N(\sigma_{\mathsf{N}}/s_0)^m}{N}\right]^N \quad \underset{N \to \infty}{\Rightarrow} \quad 1 - \mathrm{e}^{-N(\sigma_{\mathsf{N}}/s_0)^m} \tag{20}$$

because $\lim_{N\to\infty}(1 + z/N)^N = \mathrm{e}^z$. The statistics of Weibull distribution are well known and give the following large-size asymptotic properties:

$$[\bar{\sigma}_{\mathsf{N}}]_{N\to\infty} = N^{-1/m} s_0 \Gamma(1 + 1/m) \tag{21}$$

$$[\omega]_{N\to\infty} = \omega_\infty = \sqrt{\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1} \tag{22}$$

VII. Size Effect on Mean and Coefficient of Variation via Asymptotic Matching

For the mean, $\bar{\sigma}_{\mathsf{N}}$, we have the value and slope with respect to N for $N = 1$, and the slope and vertical axis intercept for asymptote at $N \to \infty$. This is a total of 4 parameters, and so the asymptotic matching formula connecting these extremes can have 4 free parameters. A suitable formula of this kind has been systematically derived,[10] and has the form:

$$\bar{\sigma}_{\mathsf{N}} = \left[\frac{N_a}{N} + \left(\frac{N_b}{N}\right)^{r/m}\right]^{1/r} \tag{23}$$

in which m, r, N_b, N_b = constants to be found from 4 matching conditions. Matching σ_{N} and $\mathrm{d}\sigma_{\mathsf{N}}/\mathrm{d}N$ to equations (11) and (17), one finds that the m-value must be the same as the Weibull modulus of the material, coinciding with the exponent of the power-law tail of RVE strength distribution. For the remaining constants r, N_a, N_b, one gets the following three equations:

$$\left[\bar{\sigma}_{\mathsf{N}}\right]_{N=1} = A = \left(N_a + N_b^{r/m}\right)^{1/r} \tag{24}$$

$$\left[\frac{\mathrm{d}\bar{\sigma}_{\mathsf{N}}}{\mathrm{d}N}\right]_{N=1} = B = -\frac{1}{r}\left(N_a + N_b^{r/m}\right)^{1/r\,-\,1}\left(N_a + \frac{r}{m}N_b^{r/m}\right) \tag{25}$$

$$\left[\bar{\sigma}_{\mathsf{N}}N^{1/m}\right]_{N\to\infty} = N_b^{1/m} = s_0\Gamma(1+1/m) \tag{26}$$

where it is assumed that, in agreement with all the experience, $r/m < 1$. Eq. (26) may now be substituted into (24) and (25). Then Eq. (24) may be solved for N_b and substituted into Eq. (25). This yields one transcendental equation for exponent r, which can be easily solved by Newton iterations, upon which N_b and N_a can be simply evaluated.

For the coefficient of variation, ω, the asymptotic matching formula is similar to Eq. (23) but ω becomes constant for large N. Therefore,

$$\omega^2 = {\omega_\infty}^2\left(1 + \frac{qN_c}{N}\right)^{1/q} \tag{27}$$

where q, ω_∞, N_c are 3 constants to be found by matching the asymptotic properties. From Eq. (22) one obtains ω_∞, and matching Eq. (17) one gets:

$$[\omega^2]_{N=1} = G = {\omega_\infty}^2\,(1+qN_c)^{1/q} \tag{28}$$

$$\left[\mathrm{d}\omega^2/\mathrm{d}N\right]_{N=1} = H = -{\omega_\infty}^2 N_c\,(1+qN_c)^{1/q\,-\,1} \tag{29}$$

Solving N_c from Eq. (28) and substituting it into Eq. (29) yields a transcendental equation for exponent q, which may be easily solved by Newton iteration. N_c then follows from Eq. (28).

VIII. Grafted Weibull-Gaussian Strength Distribution for Any Size

As shown in detail in Ref. 6(Eqs. 50-65), the cdf of structure strength may be approximated by a Weibull cdf grafted from the left onto a Gaussian (normal) cdf.[6] The graft ensures continuity of cdf and its slope, and the grafted distribution is rescaled horizontally and vertically to be normalized. According to Eq. (4), if the strength of each link in a chain is Weibullian up to stress σ_W, corresponding to link failure probability $P_1(\sigma_W)$, then the cdf of the strength of the whole chain is Weibullian for all $P_f \le P_{N_g} = 1 - [1 - P_1(\sigma_g)]^N$.

In the remaining part for $P_f > P_{N_g}$, the cdf at increasing σ closely approaches the Gaussian distribution in N is small, but the Gumbel distribution if $N \to \infty$. However, for large N the Weibullian part occupies almost the entire cdf (i.e., $P_{N_W} \rightarrow 1$), and so the Gumbel part is irrelevant.

Hence, we may assume that, approximately, the strength cdf for any N is a graft of Weibull cdf onto a Gaussian cdf, with the grafting point given approximately by

$$P_{N_g} \approx 1 - (1 - P_{1_g})^N, \qquad P_{1_g} = P_1(\sigma_g) \tag{30}$$

If $P_{N_g}, \sigma_{\mathsf{N}}, \omega$ are known for any N, the grafted Weibull-Gaussian cdf can be constructed as shown before.[6] The entire cdf being known, one can then calculate the load for which the structural failure probability is, e.g, 10^{-6}. Integrating this cdf with the pdf of the load, one can also obtain the structural failure probability for a given distribution of the load.

IX. Size Effect on Structure Lifetime

All of the foregoing analysis applies for constant temperature and a fixed load duration τ. We will now explore the question of size dependence of lifetime τ of quasibrittle structures under a given nominal stress σ_{N}.

The transition state theory with the concept of activation energy was used[7] to show that the left tail of the cdf of the strength of interatomic bonds must have the form $F(\sigma) = (C_b \kappa / kT)\sigma$ = power function of stress σ with exponent 1; where T = absolute temperature, k = Boltzmann constant, κ = coefficient of linear dependence of activation energy on σ, and C_b = constant. Based on this fact it is further shown that, for various (but constant) T and τ, the nominal strength σ_{N}/s_0 in the argument of failure probability P_1 of one RVE must be replaced by $\sigma_{\mathsf{N}}/s_0 R(\tau, T)$ where

$$R(\tau, T) \;=\; \frac{\Lambda(\tau_0)}{\Lambda(\tau)}\,\frac{T}{T_0}\;\mathrm{e}^{\left(\frac{1}{T} - \frac{1}{T_0}\right)\frac{Q}{k}} \tag{31}$$

where T_0, τ_0 = reference values of T and τ, for which $R = 1$, and function Λ indicates how the stress for which the atomic thermal vibrations produce a contiguous surface of a break in the material nanostructure scales with the load duration[7] (where, for the sake of simplicity, one may set $\Lambda(\tau) \approx r\tau$ = linear function of τ; r = constant). Hence, for a finite chain of RVEs, Eq. (4) must be generalized as

$$P_N(\sigma_{\mathsf{N}}, \tau, T) = 1 - \{1 - P_1[\sigma_{\mathsf{N}}/s_0 R(\tau, T)]\}^N \tag{32}$$

¿From this, we can obtain not only the pdf of structural strength as a function of applied

σ_N, $p_f(\sigma_\mathsf{N}) = [\mathrm{d}P_N/\mathrm{d}\sigma_\mathsf{N}]_{T,\tau}$, but also the pdf of strength as a function of load duration τ:

$$p_N(\tau) = [\mathrm{d}P_N/\mathrm{d}\tau]_{T,\sigma_\mathsf{N}} \tag{33}$$

which can be calculated from Eqs. (32) and (31). This distribution agrees with the requirements that, for $\sigma_\mathsf{N} \to 0$, one must have $P_N = 0$ and $\tau \to \infty$; while for $\sigma_\mathsf{N} \to \infty$, one must have $P_N = 1$ and $\tau \to 0$.

The mean structural lifetime (or durability) as a function of σ_N and T may be calculated as

$$\begin{aligned}
\bar{\tau}(N, \sigma_\mathsf{N}, T) &= \int_0^\infty \tau \frac{\mathrm{d}P_N}{\mathrm{d}\tau}\,\mathrm{d}\tau = \int_0^1 \tau\;\mathrm{d}P_N = \int_0^\infty (1 - P_N)\;\mathrm{d}\tau \\
&= \int_0^\infty \{1 - P_1[\sigma_\mathsf{N}/s_0 R(\tau, T)]\}^N \mathrm{d}\tau \qquad (34)\\
\text{and for } N \to \infty\text{:}\quad \bar{\tau}(N, \sigma_\mathsf{N}, T) &= \int_0^\infty \mathrm{e}^{-\;N\;P_1[\sigma_\mathsf{N}/s_0 R(\tau,T)]}\;\mathrm{d}\tau \qquad (35)
\end{aligned}$$

The last equation follows by setting $P_1 = x/N$ and noting that $\lim_{N\to\infty}(1-x/N)^N = \mathrm{e}^{-x} = \mathrm{e}^{-NP_1}$. If we assume that R is linear in τ, i.e. $\Lambda(\tau) = r\tau$, and note that $\Gamma(1/m)/m = \Gamma(1 + 1/m)$, then

$$\begin{aligned}
\text{for } N \to \infty\text{:}\quad \bar{\tau}(N, \sigma_\mathsf{N}, T) &= \int_0^\infty \mathrm{e}^{-N(\sigma_\mathsf{N}\tau/s_0\tau_0 R_0)^m} \mathrm{d}\tau \;=\; \tau_0\,\frac{s_0}{\sigma_\mathsf{N}}\,\frac{R_0(T)}{N^{1/m}}\,\Gamma\left(1 + \frac{1}{m}\right) \qquad (36)\\
\text{where}\quad R_0(T) &= \frac{T_0}{T}\,\mathrm{e}^{\left(\frac{1}{T} - \frac{1}{T_0}\right)\frac{Q}{k}} \qquad (37)
\end{aligned}$$

Note that the mean lifetime for $N \to \infty$ decreases as a power law of structure size and is inversely proportional to the applied stress. For finite N, however, there is a deviation from the power law.

For the coefficient of variation ω_τ of structural lifetime under load σ_N and at temperature T, one has

$$\begin{aligned}
{\omega_\tau}^2 &= \frac{1}{[\bar{\tau}(\sigma_\mathsf{N}, T)]^2} \int_0^1 [\tau(\sigma_\mathsf{N}, T)]^2\;\mathrm{d}P_N[R(\tau, T)\sigma_\mathsf{N}] \;-\; 1 \qquad (38)\\
\text{for } N \to \infty\text{:}\quad {\omega_\tau}^2 &= \frac{\Gamma(1 + 2/m)}{\Gamma^2(1 + 1/m)} - 1 \;=\; {\omega_\infty}^2 \qquad (39)
\end{aligned}$$

This is the same coefficient of variation as for the strength at fixed load duration, and is governed solely by Weibull modulus.

To obtain explicit approximations of $\bar{\tau}$ and ω_τ, asymptotic matching similar as before may be used. For small N, the cdf of τ is again Gaussian, except for the tail of probability $<$ circa 0.001. With increasing N, a Weibull tail grows into the Gaussian core until, for $N >$

circa 5000, it occupies essentially the entire cdf of τ.

X. Closing Comment

The mathematical model established in this article for the effect of structure size and geometry on the probability distribution of structural lifetime at constant load is not yet verified experimentally. The verification and calibration is left for a forthcoming journal article.

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References

[1]Duckett, W. "Risk analysis and the acceptable probability of failure." *The Struct. Engr.* 2005, pp. 25–26.

[2]Melchers, R.E. *Structural Reliability, Analysis & Prediction.* Wiley, New York, 1987.

[3]NKB (Nordic Committee for Building Structures). *Recommendation for loading and safety regulations for structural design.* NKB Report, No. 36, 1978.

[4]Bažant, Z.P., and Pang, S.D. "Revision of reliability concepts for quasibrittle structures and size effect on probability distribution of structural strength." *Proc., 9th Int. Conf. on Structural Safety and Reliability* (ICOSSAR), Rome, G. Augusti, G.I. Schuëller and M. Ciampoli, eds., Milpress, Rotterdam, 2005, pp. 377–386.

[5]Bažant, Z.P., and Pang, S.-D. "Effect of size on safety factors and strength of quasibrittle structures: Beckoning reform of reliability concepts." *Proc., The Structural Engineering Convention (SEC 2005)*, J.M. Chandra Kishen & D. Roy, eds., Indian Institute of Science, Bangalore, India, 2005, pp. 2–20.

[6]Bažant, Z.P., and Pang, S.-D. "Mechanics based statistics of failure risk of quasibrittle structures and size effect on safety factors." *Proc. of the National Academy of Sciences* 103 (25), 2006, pp. 9434–9439.

[7]Bažant, Z.P., and Pang, S.-D. "Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture". *J. of the Mechanics and Physics of Solids* 55, 2007, pp. 91–134.

[8]Bažant, Z.P., and Planas, J. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials.* CRC Press, Boca Raton & London 1998.

[9]Bažant, Z.P. *Scaling of Structural Strength.* Hermes, London 2002; 2nd updated ed., Elsevier, London 2005.

[10]Bažant, Z.P. "Scaling theory for quasibrittle structural failure." *Proc., National Academy of Sciences* 101 (37), 2004, pp. 14000–14007.

[11]Weibull, W. "The phenomenon of rupture in solids." *Proc., Royal Swedish Inst. Eng. Res.* **153**,

Stockholm, 1939, pp. 1–55.

[12]Ang, A.H.-S., and Tang, W.H. *Probability Concepts in Engineering Planning and Design. Vol II. Decision, Risk and Reliability.* J. Wiley, New York, 1984.

[13]Bažant, Z.P. "Size effect in blunt fracture: Concrete, rock, metal." *J. Eng. Mech.*, ASCE, **110**, 1984, pp. 518–535.

[14]Bažant, Z.P. "Scaling of quasibrittle fracture: Asymptotic analysis." *Int. J. of Fracture*, 83 (1), 1997, pp. 19–40.

[15]Bažant, Z.P. "Probability distribution of energetic-statistical size effect in quasibrittle fracture." *Prob. Eng. Mech.*, **19**, 2004, pp. 307–319.

[16]Bažant, Z.P., and Li, Yuan-Neng. "Cohesive crack with rate-dependent opening and viscoelasticity: I. mathematical model and scaling." *Int. J. of Fracture* 86 (3), 1997, pp. 247–265.

[17]Bažant, Z.P., and Novák, D. "Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. II. Application." *J. of Engrg. Mech.*, ASCE, 126 (2), 2000, pp. 175–185.

[18]Bažant, Z.P., and Novák, D. "Energetic-statistical size effect in quasibrittle failure at crack initiation." *ACI Materials Journal*, 97 (3), 2000, pp. 381–392.

[19]Bažant, Z.P., and Xi, Y. "Statistical size effect in quasi-brittle structures: II. Nonlocal theory." *J. Eng. Mech.*, ASCE, **117**, 1991, pp. 2623–2640.

[20]Bouchaud, J.-P., and Potters, M. *Theory of financial risks: From statistical physics to risk management.* Cambridge University Press, Cambridge, U.K., 2000.

[21]Curtin, W.A., and Scher, H. "Time-dependent damage evolution and failure in materials. I. Theory." *Physical Review B* 55 (18), 1997, pp. 12038–12050.

[22]Daniels, H.E. "The statistical theory of the strength of bundles and threads." *Proc. R. Soc. Lond.* **A183**, 1945, pp. 405–435.

[23]Eyring, H. "Viscosity, plasticity and diffusion as examples of absolute reaction rates." *J. of Chemistry and Physics* 4, 1936, p. 263.

[24]Fisher, R.A., Tippett, L.H.C. "Limiting forms of the frequency distribution of the largest and smallest member of a sample." *Proc. Cambridge Philosophical Society*, 24, 1928, pp. 180–190.

[25]Danzer, R. and Lube, T. "New fracture statistics for brittle materials." *Frac. Mech. Cer.*, **11**, 1996, pp. 425–439.

[26]Glasstone, S., Laidler, K.J., and Eyring, H. *The Theory of Rate Processes.* McGraw-Hill Book Co., New York 1941.

[27]Harlow, D.G., Smith, R.L. and Taylor, H.M.. "Lower tail analysis of the distribution of the strength of load-sharing systems." *J. of Applied Probability*, 20, 1983, pp. 358–367.

[28]Harlow, D.G., and Phoenix, S.L.. "The chain-of-bundles probability model for the strength of fibrous materials." *Journal of Composite Materials*, 12, 1978, pp. 195–334.

[29]Lu, C., Danzer, R. and Fischer, F.D. "Influence of threshold stress on the estimation of the Weibull statistics." *J. Am. Ceram. Soc.*, 85 (6), 2002, pp. 1640-1642.

[30]McClintock, A.M., and Argon, A.S., eds. *Mechanical Behavior of Materials.* Addison-Wesley, Reading, Mass. 1966.

[31]Mahesh, S., Phoenix, S.L., and Beyerlein, I.J.. "Strength distributions and size effects for 2D and 3D composites with Weibull fibers in an elastic matrix." *Int. J. of Fracture*, 115, 2002, 41–85.

[32]Mayer, J.E. *Statistical Mechanics.* J. Wiley & Sons, New York 1940.

[33]Newman, W.I., and Phoenix, S.L. "Time-dependent fiber bundles with local load sharing." *Physical Review E* 63, 2001, pp. 021507-1—012507-20.

[34]Phoenix, S.L. "The asymptotic time to failure of a mechanical system of parallel members." *SIAM J. on Appl. Mathematics* 34 (2), 1978, pp. 227–246.

[35]Phoenix, S.L., and I.J. Beyerlein. "Distribution and size scalings for strength in a one-dimensional random lattice with load redistribution to nearest and next nearest neighbors." *Physical Review E*, 62 (2), 2000, pp. 1622–1645.

[36]Phoenix, S.L., Ibnabdeljalil, M., Hui, C.-Y. "Size effects in the distribution for strength of brittle matrix fibrous composites." *Int. J. of Solids and Structures*, 34 (5), 1997, 545–568.

[37]Phoenix, S.L., and Smith, R.L. "A Comparison of probabilistic techniques for the strength of fibrous materials under local load-sharing among fibers", *Int. J. of Solids Structures*, 19 (6), 1983, pp. 479–496.

[38]Phoenix, S.L., and Tierney, L.-J. "A statistical model for the time dependent failure of unidirectional composite materials under local elastic load-sharing among fibers." *Engrg. Fracture Mechanics* 18 (1), 1983, pp. 193-215.

[39]Smith, R.L., and Phoenix, S.L. "Asymptotic distributions for the failure of fibrous materials under series-parallel structure and equal load-sharing", *J. of Applied Mech.*, 48, 1981, pp. 75–81.

[40]Zhurkov, S.N. "Kinetic concept of the strength of solids." *International Journal of Fracture Mechanics*, 1 (4), 1965, pp. 311–323.

[41]Zhurkov, S.N., and Korsukov, V.E. "Atomic mechanism of fracture of solid polymers." *Journal of Polymer Science*, 12 (2), 1974, pp. 385–398.