Scaling of Failure Probability of Quasibrittle Structures with Large Cracks

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ABSTRACT: In the present paper, a probabilistic generalization of the Type 2 size effect is proposed. The generalization is derived through a large size asymptotic expansion of LEFM equations combined with a weighted finite weakest-link model, which is justified by the fact that the structure survives if the crack tip location in any of a finite number of representative volume elements (RVE) cannot cause failure. Compared to Type 1 failure, the zone of possible crack tip locations of non-negligible probability is much smaller and its extent in dimensionless coordinates decreases as the structure size increases. Consequently, the statistical component of the Type 2 size effect on the mean structural strength is much weaker. Yet, the size has a significant effect on the coefficient of variation of structural strength. An application to the probability distribution of the diagonal shear failure of reinforced concrete beams is demonstrated.

1 INTRODUCTION
Quasibrittle materials represent a broad and increasingly important class of materials. Although the constituents of quasibrittle structures are brittle, a heterogeneous microstructure causes the RVE not to be negligible compared with the characteristic size $D$ (or cross-section dimension) of the structure. Depending on the scale of observation or application, quasibrittle materials include concrete, fiber composites, toughened ceramics, rigid foams, nanocomposites, sea ice, consolidated snow, rocks, mortar, masonry, fiber-reinforced concretes, stiff clays, silts, grouted soils, cemented sands, wood, paper, particle board, filled elastomers, various refractories, coal, dental cements, bone, cartilage, biological shells, cast iron, and many modern tough alloys.

A realistic quasibrittle structural design across various engineering fields urgently needs the development of effective scaling laws for fracture and their theoretical justification (Bažant, 2002). As is now generally accepted, two types of size effect on the strength of quasibrittle structures must be distinguished: Type 1 or 2 size effects occur in structures failing either at initiation of a macroscopic crack from a smooth surface or after large stable crack growth. The laws of mean size effect for both types were derived deterministically, by asymptotic expansion of energy release according to the equivalent linear elastic fracture mechanics (LEFM) (Bažant, 2002, Bažant, 2004).

However, the mean size effect is only one aspect of the problem. In mechanical design as well as hazard mitigation, one must ensure an extremely low failure probability such as $10^{-6}$. To achieve this goal the pdf of structural strength must be fully determined (Bažant & Pang, 2006). For the Type 1 size effect, a probabilistic generalization was obtained by means of the finite weakest-link model and scale transition from failure probability on the nanoscale (Bažant & Pang, 2006, Bažant et al. 2008, Bažant & Le 2009). However, a similar general study on Type 2 size effect is still lacking.

The purpose of this paper is to present a probabilistic generalization of the Type 2 size effect obtained by deterministic fracture analysis. The generalization is derived by a large-size asymptotic expansion of LEFM combined with a weighted finite weakest-link model. The approach is justified by the fact that the structure survives if crack tip locations in any of a finite number of representative volume elements (RVE) cannot cause failure. The elements of the weakest-link model are weighted by the energy release rates calculated for large crack tip locations in various RVEs by means of a finite element model. Compared to Type 1 failure, the zone of possible crack tip locations of non-negligible probability is much smaller and its extent in dimensionless coordinates decreases as the structure size increases. Consequently, the statistical component of the Type 2 size effect on the mean structural strength is much weaker. However, the size has a significant effect on the coefficient of variation of structure strength. An
application to the probability distribution of the diagonal shear failure of reinforced concrete beams is demonstrated.

2 THEORETICAL FORMULATION

2.1 Structural strength for a particular crack tip position

Like in all studies of strength statistics, it is assumed that the tip of the dominant crack may lie, at the maximum load state, at various locations—particularly at the centers of square elements \( i = 1, 2, \ldots N_D \) in a square grid, as exemplified for shear failure of a reinforced concrete beam in Fig. 1. Each square element is assumed to represent an RVE of the material of characteristic size \( l_0 \) (which implies an autocorrelated random strength field). Based on Bažant’s size effect law (Bažant & Planas 1998, Bažant. 2002) it can be shown that:

\[
\sigma_{Ni\beta} = \frac{B_{i\beta}}{\sqrt{1 + D / D_{0i\beta}}} f_i' \tag{1}
\]

where \( D_{0i\beta} = \frac{g_{i\beta}}{g_{i'\beta}} c_f \)

\[
B_{i\beta} = \sqrt{\frac{l_0}{g_{i'\beta} c_f}}, \quad l_0 = \frac{E' G_f}{f_i'} \tag{2}
\]

Here \( F \) = given load (maximum load), \( b \) = beam thickness, \( E \) = elastic modulus, \( G_f \) = fracture energy, \( l_0 \) = Irwin’s characteristic material length roughly equal to the length of the FPZ, \( c_f \) = material characteristic length for size effect (\( c_f / l_0 \approx 0.44 \) for 3PB tests Cusatis & Schaufert, 2009); \( g_{i\beta} \) = dimensionless energy release rate for crack extension from element \( i \) in the direction \( \beta \), and \( g_{i'\beta} \) = derivative with respect to the crack extension length in direction \( \beta \); and \( \sigma_{Ni\beta} \) = nominal strength of structure of size \( D \) when the failure is due to crack extending in direction \( \beta \) from the \( i \)-th RVE with tensile strength \( f_i' \). This means that the crack growth is considered to be governed by the cohesive crack model or crack band model, or some nonlocal damage model, in which \( f_i' \) is one basic material fracture characteristic, which is considered random). Before undertaking the failure analysis, the values of \( g_{i\beta} \) and \( g_{i'\beta} \) are evaluated by J-integrals and finite elements for different directions \( \beta \) of crack propagation. The direction \( \beta \) that gives the overall maximum energy release rate is determined and fixed (for the sake of simplicity, with same value for all tips \( i \)). The energy release rate also depends on the shape of the crack (which itself is a result of Markov random process of crack growth) and on the location of its starting point, but these effects appear to be secondary and are here neglected, for the sake of simplicity.

What greatly simplifies the problem is the fact that the deterministic size effect law in Eq. (1), which is already well established, can be imposed as the input, rather than being solved as part of the analysis. It is through this law that the energetic aspects of fracture mechanics are conveniently introduced.

2.2 Grafted Probability Distribution for one RVE

The probability that the structure of size \( D \) fails due to a crack extending in direction \( \beta \) from the \( i \)-th RVE of tensile strength \( f_i' \) may be written as

\[
P_{\beta i} = \text{Prob}(\sigma_{Ni\beta} < \sigma_{ND}^l) \quad \text{where} \quad \sigma_{ND}^l = \frac{F}{b D} \tag{4}
\]

where \( \sigma_{ND}^l \) is the nominal stress of structure under given load \( F \). Then, since:

\[
P_{\beta i} = \text{Prob} \left( f_i' \leq \frac{\sqrt{1 + D / D_{0i\beta}}}{B_{i\beta}} \sigma_{ND}^l \right) \tag{5}
\]

the following basic statistical hypothesis now appears logical:

\[
P_{\beta i} = \Phi_{GW} \left( \frac{\sqrt{1 + D / D_{0i\beta}}}{B_{i\beta}} \sigma_{ND}^l \right) \tag{6}
\]

where \( \Phi_{GW} \) = cumulative probability distribution function (cdf) characterizing the tensile strength of one RVE. Based on previous work (Bažant, 2007, Le et al. 2011), it must have the form of a grafted Gauss-Weibull probability distribution, which was derived theoretically from nano-mechanics and multiscale transition of probability tail, and was extensively verified and calibrated by Type 1 tests of size effect and histograms.

2.3 Determination of the failure probability

In deterministic analysis, only one specific crack tip location corresponds to failure. But if the material is random, every location could correspond to failure, albeit with a very different probability \( P_{\beta i} \). For a random material, the structure of size \( D \) will survive under load \( F \) if none of the crack tips \( i \) leads to failure. So, according to the joint probability theorem:

\[
1 - P_{\beta i} = \prod_{i=1}^{N_D} \left[ 1 - \Phi_{GW} \left( \frac{\sqrt{1 + D / D_{0i\beta}}}{B_{i\beta}} \sigma_{ND}^l \right) \right] \tag{7}
\]

where \( P_{\beta i} = \) failure probability of the structure of size \( D \); and \( \Phi_{GW} = \) cumulative grafted Gauss-
Weibull probability distribution of the strength of one RVE of the material, derived theoretically from nano-mechanics and multiscale transition of probability tail, and calibrated experimentally by Type 1 size effect tests. The point to note is that, like in Type 1 strength analysis, the number of independently contributing material elements is greater in a large structure than it is in a small one; see Fig. 2. This intuitively explains the cause of the statistical size effect in Type 2 failures. What allows us to base the failure probability analysis on the strength (rather than the critical energy release rate) of one RVE is that fracture of a quasibrittle material is properly analyzed in terms of the cohesive crack model or crack band model, for which the material strength is one essential material property characterizing the distributed damage in the FPZ. The lecture presents and reviews the results of this analysis for typical reinforced concrete beams of various sizes, including the statistical effect on the mean nominal strength and on the coefficient of variation of nominal strength.

3 APPLICATION OF THE THEORY TO TYPE 2 FAILURE IN BEAM SHEAR

The proposed theoretical formulation is applied to the shear failure of a beam, which is known to develop a large crack before failure. The analyzed case refers to the investigations by Ozbolt J. & Elighausen, (1991) where four geometrically similar specimens based on experiments conducted by Bazant & Kazemi (1991) are analyzed using the non-local microplane model and four-node plane stress finite elements. The average scale-independent crack path has been derived from the experimental data such as that of Kani (1967) on specimens of different sizes.

To find the structural strength related to each crack tip location by means of Eq. (2), the dimensionless strain energy release $g(\alpha)$ and its derivative $g'(\alpha)$ must be calculated for each crack configuration. In this study, a numerical model has been implemented by means of the commercial code ABAQUS®. Eight node quadrilateral elements have been used for the computations. The strain energy release has been computed by $J$-integral calculations using a fine mesh oriented around the crack tip. For each configuration, the direction of crack propagation has been assumed according to a maximum strain energy release condition (Hussain et al., 1974). A beam of a particular size was used to perform the finite element calculations and the resulting quantities could then be non-dimensionalized in order to utilize them in calculations of beams of any size. While the autocorrelation length, $l_0$, is the same for different sizes, the ratio of the correlation length to the size of the structure varies. As a result, the resulting field of $g(\alpha)$ and $g'(\alpha)$ values were used to compute the strength distribution of different size specimens by scaling the selection of points as illustrated in Figure 2.

The predicted values of structural strength related to each RVE can be computed by means of Eq. (2) for any specimen size. These results are then used to estimate the number of RVEs which contribute the most in the process of failure and which should be taken into account when invoking the joint probability theorem.

4 RESULTS AND DISCUSSION

In the previous sections, a general methodology for the probabilistic analysis of Type II size effect was outlined. The approach was founded on the application of the mean size effect law and the finite weakest link model to derive a general probabilistic description of Type II size effect. As a general example, the model was applied to the diagonal shear failure of reinforced concrete beams. Preliminary results indicated a systematic decrease in the coefficient of variation of strength with increasing specimen size. Since the location of fracture initiation is predetermined to occur at the crack tip and...
thus cannot sample the random strengths of many different RVEs, the statistical contribution to the mean size effect is found to be negligible.

5 CONCLUSIONS

In the present paper, a probabilistic generalization of the Type 2 size effect is proposed. A large size asymptotic expansion of LEFM combined with a finite weighted weakest-link model is used in the analysis. This approach agrees with the assumption that the structure survives if a crack tip located in any of a finite number of representative volume elements (RVE) cannot cause failure.

An application to the probability distribution of the diagonal shear failure of reinforced concrete beams is demonstrated. It is shown that, compared to the Type I case, the statistical component of the Type 2 size effect on the mean structural strength is relatively weak. However, the size has a significant effect on the coefficient of variation of structure strength.

Final comment: The results of computer analysis must be subjected to various safety factors, particularly the overload factors and the understrength factors dictated by the design code. At present, these factors are mostly empirical and highly uncertain, mainly because they do not properly reflect the tails of probability distributions and the size effect. Thus, in fact, the design code provisions accounting for uncertainty are more uncertain than anything else in design. Therefore, to benefit from accurate computer analysis of structures, these tails and the size effect must be realistically incorporated into the system of safety factors. Until that happens, a highly accurate computer simulation of the strength of concrete structures has only little practical value.

REFERENCES


