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CONSTITUTIVE LAWS FOR ENGINEERING MATERIALS

RECENT ADVANCES AND INDUSTRIAL AND INFRASTRUCTURE APPLICATIONS

RECENT PROGRESS IN DAMAGE MODELING: NONLOCALITY AND ITS MICROSCOPIC CAUSE

Zdeněk P. Bažant
Walter P. Murphy Professor of Civil Engineering
Northwestern University, Evanston, IL 60201, USA

ABSTRACT

The lecture has a twofold aim: To briefly review the current status of nonlocal modeling of damage, and to present a new justification of the nonlocality of strain-softening damage due to microcracking. The continuum damage due to microcracking is analyzed assuming a simplified micromechanics model in which interactions of adjacent microcracks are neglected and the microcracks are arranged at the centers of the cells of a cubic mesh. The release of stored energy caused by the formation of one microcrack is calculated as a function of the associated relative displacement across the cell, which corresponds to the average strain of the macroscopic continuum. After imposing the homogenization conditions of equal energy dissipation and of displacement compatibility between the micromechanics model and the macroscopic continuum, it is shown that damage is a nonlocal variable that is a function of the averaged (nonlocal) strain from a certain neighborhood of the given point. This corroborates the hypothesis of nonlocal damage, which was originally introduced for other reasons. The cause for the nonlocality of damage is that whether a microcrack will form depends on the energy stored in a finite region of nonnegligible size around the potential microcrack.

PART I. COMMENTS ON CURRENT STATUS OF NONLOCAL MODELS

A continuum with nonlocal damage has recently been shown to be an effective approach for the analysis of strain-softening structures (Bažant and Pijaudier-Cabot 1987; Pijaudier-Cabot and Bažant 1987; Bažant, Lin and Pijaudier-Cabot 1987; Bažant and Pijaudier-Cabot 1988; Bažant and Lin 1988a,b). The basic idea of the nonlocal continuum model is that only the damage is nonlocal, being a function of the strain average from a certain neighborhood of a given point, while all the other variables, especially the elastic strain, are local.

By contrast, in the original nonlocal continuum models for elastic materials (Kröner 1968; Krumhansl 1968; Kunin 1968; Levin 1971; Eringen and Edelen 1972; Eringen and Ari 1983; etc.), as well as in the first nonlocal model for strain-softening continuum (Bažant 1984; Bažant, Belytschko and Chang 1984), the elastic strain and total strain were nonlocal. This led to certain numerical difficulties (Bažant and Pijaudier-Cabot 1988), for example the existence of spurious zero-energy instability modes (which had to be suppressed artificially by overlay with local continuum), the presence of spatial integrals or higher-order instability modes (which had to be suppressed artificially by overlay with local continuum), the presence of spatial integrals or higher-order derivatives in the differential equations of equilibrium or motion and in the boundary and interface conditions, and an imbricate structure of the finite element approximation which however proved cumbersome for programming.

These difficulties were later shown to be a consequence of imposing symmetry on the integral or differential operators involved. The symmetry is lost with the nonlocal damage concept, which means that the tangential (but not the elastic) structural stiffness matrix of the finite element approximation is nonsymmetric (Bažant and Pijaudier-Cabot 1988). But the particular type of nonsymmetry obtained does not appear to cause any numerical difficulties, even for strain-softening structures with thousands of nodal displacements (Bažant and Lin 1988a).

The nonlocal concept eliminates the problems with spurious mesh sensitivity and incorrect convergence. It assures that refinements of finite element mesh cannot lead to spurious localization of strain, damage and energy dissipation into a strain-softening zone of vanishing volume. The most important physical property of nonlocal continuum damage is that for geometrically similar structures it yields a size effect that is transitional between plasticity (no size effect) and linear elastic fracture mechanics (the strongest possible size effect). This size effect is evidenced by extensive laboratory measurements on various kinds of concrete structures and fracture specimens of concrete and rock, as well as the available test data for fracture of ice (e.g. Dempsey 1990) and toughening ceramics (cf. Bažant and Kazemi 1990, with further references). The finite element codes based on local continuum cannot capture the size effect, which is a major fault when structures with damage are analyzed. Correct modeling of the experimentally observed size effect should be adopted as the basic criterion of acceptability of any finite element code for concrete structures or rock.

Physical justification by micromechanics, however, has been rather limited. In a recent study (Bažant 1987) it was suggested that the physical source of nonlocality of damage is the fact that the formation and growth of a microcrack depends on the strain energy stored in a nonzero volume of the material surrounding the microcrack, whose release drives the growth of the microcrack. Considering a quasiperiodic microcrack array and analysing the displacements due to fracture, it was shown that, under certain simplifying assumptions, the damage is a function of the of the spatially averaged fracturing strain of the macroscopic smoothing continuum, which implies damage to be nonlocal. This form of damage, however, does not seem to be the most convenient formulation, and does not quite agree with the nonlocal damage formulations used in the above-mentioned finite element models.

Aside from presenting a brief review of the advances amd problems just described, the lecture has the secondary objective of presenting the following new justification of the nonlocality of continumum damage that is due to a system of densely distributed microcracks.

PART II. JUSTIFICATION OF NONLOCALITY OF DAMAGE DUE TO MICROCRACKS

Consider an elastic material with penny-shaped microcracks of various diameters 2a. We imagine the material to be subdivided into cubical cells of side ℓ (Fig. 1a), each of which contains approximately in the middle one microcrack. For the sake of simplicity, we suppose each microcrack to be sosmall ($a \ll \ell$) that its interaction with other microcracks, as well as the energy release from the adjacent cells, is negligible.

We analyze one microcrack and allign the cell so that its one side as well as the coordinate axis x be parallel to this microcrack (Fig. 1a,b). We suppose that the microcrack plane is normal to the maximum principal stress at the center of the cell before cracking, denoted as σ, and for the sake of simplicity we assume that the normal strains in the direction parallel to the crack are constant, as illustrated by imagined sliding restraints on the sides of the cell shown in Fig. 1b. We assume the variation of σ over the cell to be sufficiently small, so that the stress intensity factor K of the microcrack is approximately the same as that for a penny-shaped crack in an infinite elastic solid with stress σ at infinity, which is as follows (cf. Broek 1986; Knott 1973; Tada et al. 1985; Murakami 1987)

$$K_I = 2\,\sigma\,\sqrt{a/\pi} \tag{1}$$

We now try to calculate the energy release due to crack formation as a function of the deformation of the cell. We begin by writing the rate of release of energy (complementary energy) W_f^* due to fracture:

$$\partial W_f^* \,/\, \partial a = 2\,\pi\,a\,K_I^2 \,/\, E' = 8\,\sigma^2 a^2 / \,E' \tag{2}$$

where $E' = E/(1-\nu^2)$, E = Young's elastic modulus, ν = Poisson ratio. Since the material is elastic and thus path-independent, we may consider for the purpose of energy calculation that the crack has formed under constant stress σ. Then, by integration of Eq. 2, the total energy release caused by the microcrack is obtained as

$$W_f^* = 8\,a^3\sigma^2 / \,3\,E' \tag{3}$$

Let δ be the total relative displacement between the opposite sides of the cell ($\delta = u_2 - u_1$ where u_2, u_1 = displacements in the x-direction at the opposite sides of the cell) and δ_f the relative displacement due to crack formation, which is approximately equal to the relative displacement in an infinite solid between its opposite infinities. In the diagram of $\sigma\ell^2$ (the force acting on the sides of the cell) versus δ (Fig. 2a), W_f^* is represented by the area 0120. This triangular area is equal to $W_f^* = \delta_f \sigma \ell^2/2$. Setting this equal to Eq. 3, one gets

$$\delta_f = \frac{16}{3E'}\,\frac{a^3}{\ell^2}\,\sigma \tag{4}$$

The same expression can be obtained from Eq. 3 by Castigliano's theorem, which implies that $\delta_f = \partial W_f^*/\partial(\sigma\ell^2)$.

Consider now that the crack forms at constant δ rather than at constant σ. This must be equivalent to first unloading the uncracked solid from stress σ_0 to a certain stress $\sigma_0 - \sigma_f$ (path $\overline{12}$ in Fig. 2b) and, second, letting the crack grow at constant stress (path $\overline{23}$ in Fig. 2b), provided that $\sigma_f = E'u_f/\ell$ (from triangle 123 in Fig. 2b), in order to guarantee that the displacement increase δ_f due to crack formation at constant stress $\sigma_0 - \sigma_f$ restores the original total displacement δ_f (point

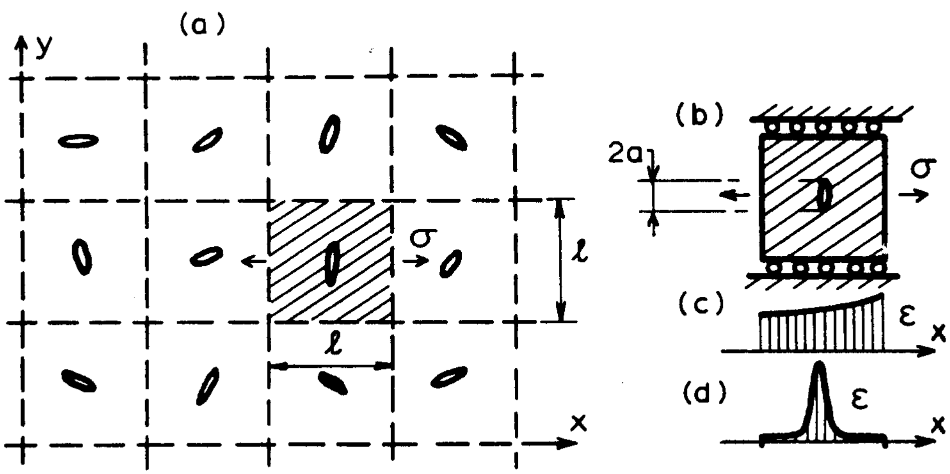

FIG. 1. (a) Array of cubical cells containing microcracks, and (b) one cell with simplified boundary conditions considered in calculations.

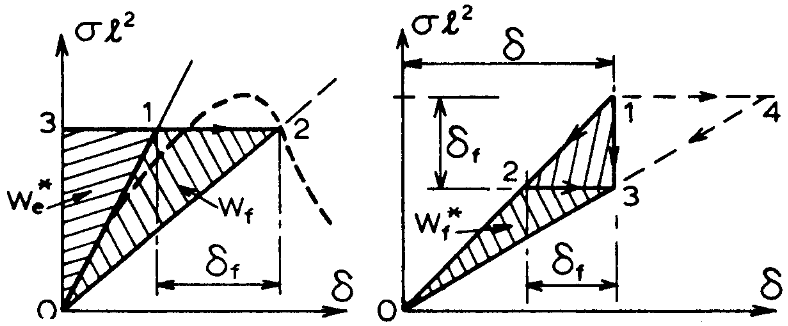

FIG. 2 Energy released due to crack formation (a) at constant stress, and (b) at constant displacement.

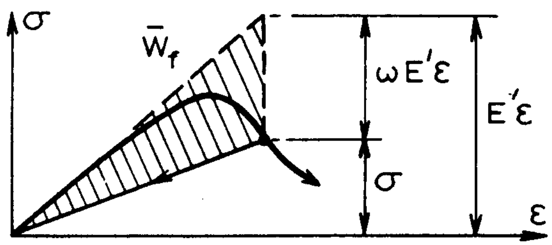

FIG. 3 Stress-strain relation of continuum damage mechanics with strain-softening and energy release.

3 in Fig. 2b). In analogy to Eq. 4, we have for this process $\delta_f/\ell = 16(a/\ell)^3(\sigma_0-\sigma_f)/3E'$ (segment $\overline{23}$ in Fig. 2c). If we set this equal to $\delta_f/\ell = \sigma_f/E'$ and solve for σ_f, we get

$$\sigma_f = \frac{\sigma_0}{1 + \frac{3}{16}\left(\frac{\ell}{a}\right)^3} \tag{5}$$

where $\sigma_0 = E'\delta/\ell$ = initial stress before cracking. The (complementary) energy released by crack formation at constant stress at $\sigma_0-\sigma_f$ is represented by area 0230 in Fig. 2b, while the energy (not complementary energy) released by crack formation at constant displacement δ is represented by area 0130 in Fig. 2b, the value of which is

$$W_f = \sigma_f \frac{\delta}{\ell}\frac{\ell^3}{2} = \frac{E' \ell \delta^3}{2 + \frac{3}{8}\left(\frac{\ell}{a}\right)^3} \tag{6}$$

Note that since the material is elastic, that is path-independent, the same result must be obtained if one considers the path 143 (instead of 123) in Fig. 2b, for which, first, the crack is formed at constant stress σ_0 and, second, the cell is unloaded so as to restore the original relative displacement δ.

For a propagating crack we must have $K_I = K_R(a)$ = given R-curve (crack resistance curve) = critical stress intensity factor required for further crack growth, which must be determined in advance. Eq. 1 with σ replaced by $\sigma-\sigma_f$ (path $\overline{23}$ in Fig. 2b) then provides

$$a = \frac{\pi K_R^2(a)}{4(\sigma_0 - \sigma_f)^2} \tag{7}$$

Substituting $\sigma_0 = E'\delta/\ell$ and Eq. 5 for σ_f, and solving the resulting equation for δ/ℓ, we acquire the relation

$$\frac{\delta}{\ell} = \frac{1}{2E'}\sqrt{\frac{\pi}{\ell}\frac{\ell}{a}}\, K_R\!\left(\ell\frac{a}{\ell}\right)\left[1 + \frac{16}{3}\left(\frac{a}{\ell}\right)^3\right] = \phi\left(\frac{\ell}{a}\right) \tag{8}$$

where ϕ is a function, defined by this relation. Denoting the inverse function as ψ (and supposing function ϕ to be invertible), we may write

$$\frac{\ell}{a} = \psi\left(\frac{\delta}{\ell}\right) \tag{9}$$

Substituting this into Eq. 6 we obtain the desired result

$$W_f = \frac{E' \ell \delta^3}{2 + \frac{3}{8}\psi^3\left(\frac{\delta}{\ell}\right)} = f\left(\frac{\delta}{\ell}\right) \tag{10}$$

where f is a function.

For other geometries of microcracks and repetitive cells, one may expect similar results, but with different expressions for functions ϕ, ψ and f.

Let ε be the (local) macroscopic strain (normal strain in the direction of principal stress σ), and $\langle\varepsilon\rangle$ the average (nonlocal) macroscopic strain, defined as $\langle\varepsilon(\underset{\sim}{x})\rangle = \ell^{-3}\int_V \varepsilon(\underset{\sim}{s})dV(\underset{\sim}{s})$ where V is the volume of the cell, $\langle\ \rangle$ denotes the spatial (nonlocal) averaging operator, $\underset{\sim}{x}$ is the coordinate vector of the center of the cell, and $\underset{\sim}{s}$ are the cordinate vectors of the points of the cell. For the sake of brevity, we will delete in the following the coordinates $\underset{\sim}{x}$ and $\underset{\sim}{s}$, simply writing

$$\langle\varepsilon\rangle = \frac{1}{\ell^3}\int_V \varepsilon\, dV \qquad (11)$$

As one homogenization condition, strains ε must be compatible with displacement δ due to crack, which is satisfied by

$$\delta = \ell\,\langle\varepsilon\rangle \qquad (12)$$

The energy released by crack formation at constant δ may now be rewritten as

$$W_f = \frac{E'\,\ell^3\,\langle\varepsilon\rangle^2}{2 + \frac{3}{8}\psi^3(\langle\varepsilon\rangle)} = f(\langle\varepsilon\rangle) \qquad (13)$$

or more precisely $W_f(\underset{\sim}{x}) = f[\langle\varepsilon(\underset{\sim}{x})\rangle]$. This equation, which shows that W_f is a function of the average (nonlocal) strain rather than the (local) strain, is the key for the nonlocal character of continuum damage. Note that simplifying Eq. 13 as a local relation $W_f(x) = f[\varepsilon(x)]$ can make a large difference if the strain distribution approaches Dirac delta function as shown in Fig. 1d, which is known to typically happen in local continuum damage formulations. In fact, the nonlocality is what enforces smooth (nonlocalized) strain distributions.

Consider now the standard stress-strain relation of continuum damage mechanics (Kachanov 1958; Lemaitre and Chaboche 1978). Its simplest form is

$$\sigma = (1-\omega)\,E'\,\varepsilon \qquad (14)$$

where ω is called the damage, supposed here to be a scalar, for the sake of simplicity. Continuum damage mechanics assumes the unloading stiffness to be given by the secant modulus, which is equal to $(1-\omega)E'$ and corresponds to line $\overline{02}$ in Fig. 3. The energy release (dissipation) due to damage is given by the area of triangle 0120 in Fig. 3, which is $\omega E'\varepsilon^2/2$. Thus the energy release from volume V of the cell is

$$W_f = \int_V \omega\,\frac{E'\varepsilon^2}{2}\,dV = \frac{E'}{2}\,\ell^3\,\langle\,\omega\,\varepsilon^2\,\rangle \qquad (15)$$

At this point, subject to further confirmation, we anticipate damage ω to be a nonlocal variable characterizing the behavior of the volume V as a whole. This permits simplifying Eq. 15 by treating ω as a constant, even though it may vary with spatial coordinates x, y, z. Accordingly, Eq. 15 may be approximately replaced by the equation

$$W_f = \frac{E'}{2}\,\ell^3\,\omega\,\langle\,\varepsilon^2\,\rangle \qquad (16)$$

As a second homogenization condition, the energy releases calculated for the cell of the micromechanics model (Eq. 10) and for volume V of the macroscopic continuum having the same center (Eq. 16) must be equal. Solving this equality for ω, we acquire the following result:

$$\omega = \frac{\langle\varepsilon\rangle^2}{\langle\varepsilon^2\rangle}\left[1 + \frac{3}{16}\psi^3(\langle\varepsilon\rangle)\right]^{-1} = \frac{2\,f(\langle\varepsilon\rangle)}{E'\,\ell^3\,\langle\varepsilon\rangle^2} = \frac{\langle\varepsilon\rangle^2}{\langle\varepsilon^2\rangle}F(\langle\varepsilon\rangle) \qquad (17)$$

in which F is a function. If the variation of ω within volume V is not too strong, one may use the approximation $\langle\varepsilon^2\rangle \simeq \langle\varepsilon\rangle^2$, and then

$$\omega = F(\langle\varepsilon\rangle) \qquad (18)$$

This confirms that damage due to microcracking is nonlocal (a result we anticipated in our previous assumption), and must be considered as a function of the average (nonlocal) macroscopic strain rather than the local macroscopic strain.

If we did not replace Eq. 15 by Eq. 16, the result would be that

$$\langle\,\omega\,\varepsilon^2\,\rangle = \hat{F}(\langle\varepsilon\rangle) \qquad (19)$$

where $\hat{F}$ is a function. This still means that damage is nonlocal, but its calculation from strains in a finite element program would be implicit, and thus more involved.

CONCLUSION

Spurious localization of strain-softening damage, along with the inherent spuriousm mesh sensitivity and nonobjectivity, can be prevented by the nolocal continuum concept. An effective and easily usable formulation results when only the strain du to microcracking is considered an nonlocal while the elastic strain is local, same as in the classical material models.

A simplified micromechanics analysis shows that continuum damage due to microcracking must be nonlocal, expressed as a function of the spatially averaged (nonlocal) strain in a certain neighborhood of the given continuum point. The reason, simply stated, is that the fracturing strain due to damage is the result of the release of stored energy from the microcrack neighborhood, the size of which is not zero but finite.

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