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LARGE-SCALE FRACTURE OF SEA ICE PLATES

ZDENĚK P. BAŽANT
Walter P. Murphy Professor

Department of Civil Engineering
Northwestern University
Evanston, Illinois 60208
USA

ABSTRACT

Propagation of bending fracture along a floating sea ice plate, representing an elastic plate on elastic foundation, is analyzed in the sense of linear elastic fracture mechanics (LEFM), assuming the entire plate thickness to be fractured all the way to the fracture tip. The analysis focuses on the size effect. Two specific problems are solved in an idealized form: steady state propagation of thermal bending fracture and vertical penetration of an object through the plate. In both problems it is found that the nominal strength (equivalent to the critical temperature difference in the first problem) decreases as $(thickness)^{-3/8}$. In the thermal problem, the $(-3/8)$-power law is exact (under the hypotheses of the analysis), while in the penetration problem it is only approximate, approached asymptotically for large sizes. This $(-3/8)$-power law differs from that known for two-dimensional or axisymmetric LEFM, in which the nominal strength scales as $(\text{size})^{-1/2}$; the difference is caused by the presence of a characteristic length for bending disturbance decay along the plate. An in-plane compressive force makes the size effect stronger than $(\text{thickness})^{-3/8}$, and an in-plane tensile force makes it weaker. Creep and propagation rate effects are approximately taken into account in the thermal problem, and so is the effect of in-plane forces on bending. A numerical example shows that the temperature changes occurring in the Arctic suffice, according to the present theory, to produce bending fractures. Such fractures might serve to initiate the formation of leads of open water, pressure ridges and rafting.

INTRODUCTION

Ice, especially sea ice, is a quasibrittle material whose failure behavior is intermediate between plasticity and linear elastic fracture mechanics (LEFM). The latter must be approached for large-scale behavior, as theory of quasibrittle fracture indicates [Bažant and Cedolin, 1991]. Fracture mechanics of ice has been studied extensively and much has already been learned [see, e.g., Sanderson 1988, Urabe and Yoshitake 1981, Bentley et al. 1988, DeFranco and Dempsey 1990, Dempsey 1989, 1990, Dempsey et al. 1989, 1990, 1991, Ketcham and Hobbs 1969, Palmer et al. 1983, Parsons 1991, Parsons et al. 1987, 1980, 1989, Stehn 1990, Timco 1991, Timco and Sinha 1988, and Weeks and Mellor 1984]. It appears, however, that most of the existing studies have been confined to the laboratory scale, from which the applicability of fracture mechanics to field problems is not directly apparent.

A low degree of notch sensitivity of laboratory specimens has been raised as an objection to applying fracture mechanics. Recently, however, Dempsey et al. [1991] showed that the notch sensitivity greatly increases with specimen size. This corroborates that indeed the applicability of fracture mechanics is a question of scale, as theory indicates. The problem has often been confused by the myth of "material brittleness". It has often been thought that a steep decline of the post-peak load deflection diagram indicates material brittleness and the occurrence of instability right after the peak load state means perfect material brittleness. These phenomena, however, cannot be material properties. Rather, they are properties of the entire structure, being governed by stability and bifurcation criteria for the structure, and change with structure size as well as shape. Furthermore, it has often been suggested that "material brittleness" is determined by the strain rate, slow deformation leading to ductile response and fast deformation to brittle response. But again, the type of response depends on the entire structure, which cannot be correctly characterized without taking creep separately into account. It is even possible that the structural response is more brittle for slow loading than fast loading (this occurs when stress relaxation in the fracture process zone is so strong that it causes this zone to shrink). Brittleness can be meaningfully, unambiguously characterized only by some measure of proximity of the response of the entire structure to LEFM. It is a structural characteristic, which of course depends on the structure size (and shape).

It might be objected that the size effect on the nominal strength of structures ought to be described by Weibull's statistical theory. Recently it has been shown, however, that this explanation is incorrect for quasibrittle materials in which large stable crack growth occurs before the maximum load [Bažant and Xi, 1990].

Strictly speaking, ice fracture should be analyzed according to nonlinear fracture mechanics in which the localization of damage and the finite volume of the fracture process zone are taken into account. However, such analysis, already accomplished for concrete, must be done numerically on a large computer. This is not only difficult but obscures the basic general properties. To bring such properties to light, we will use an analytical approach, which must of course be suitably simplified to be feasible. The simplification is achieved by adopting LEFM, which represents an extreme type of behavior. The reality in general lies between plastic limit analysis and LEFM, the latter being approached with increasing size. While plastic limit analysis (or elasticity with allowable stress) exhibits in general no size effect, LEFM exhibits the strongest possible size effect when geometrically similar situations are compared. In recognition of these facts, the size effect has currently emerged as a topic of major interest for ocean ice dynamics [Curtin, 1991].

We will focus on two important problems: (1) Initiation of long fractures that serve as the triggering events for the formation of leads of open water, pressure ridges and rafting, for which we will explore the hypothesis that they might be caused by the release of the energy of thermal bending moments, and (2) penetration (downward or upward) of an object through the floating ice plate. In this paper the analysis is presented in a rather condensed form, but the analysis of thermal bending fracture will be given in detail in Bažant [1992].

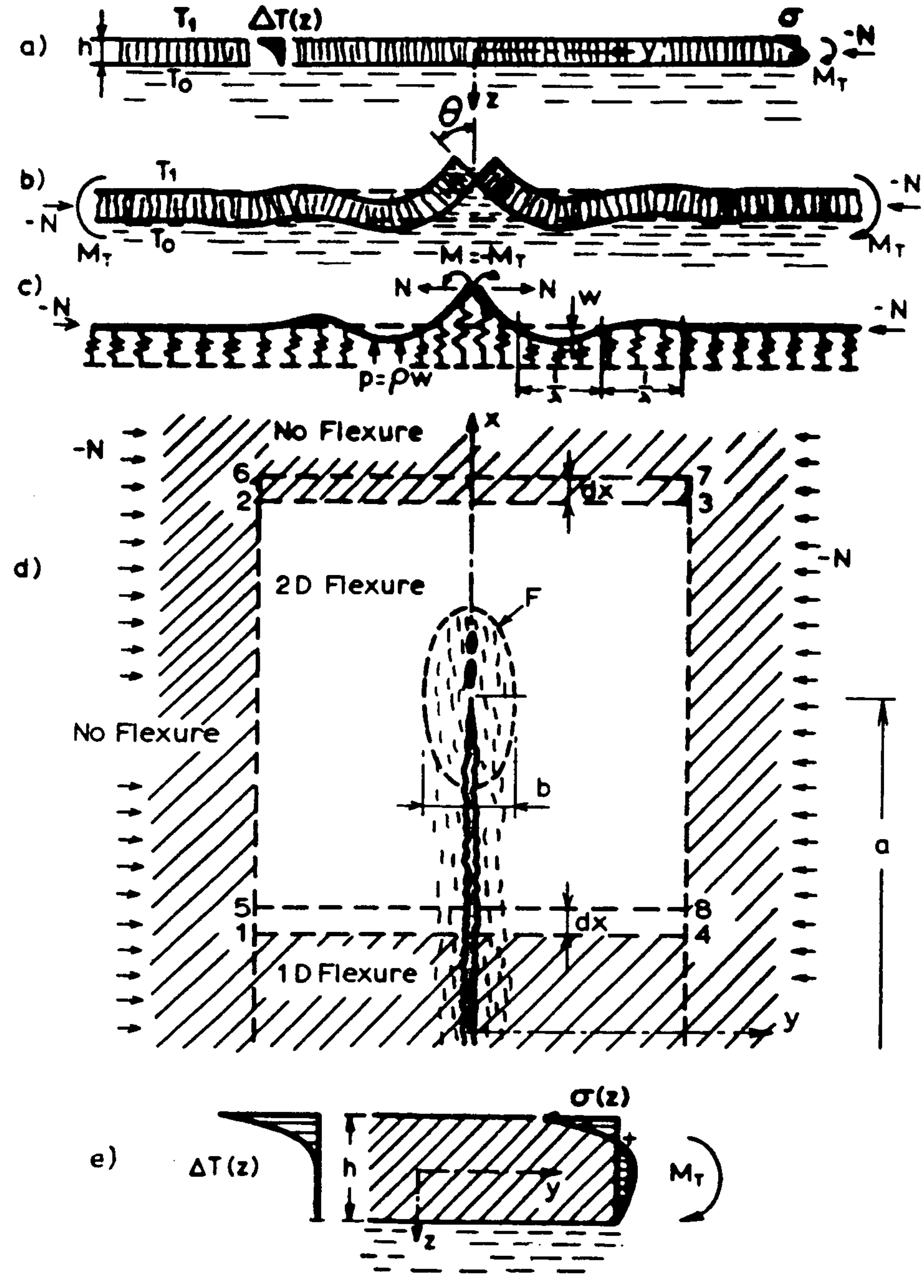

Figure 1: (a) Floating ice plate, (b) its deflections, (c) the beam on elastic foundation, (d) view of propagating fracture from top, and (e) profiles of temperature and thermal stress.

state of the plate in the region *1234* is very complex. But we do not need to know it if we study only the *steady-state* crack propagation. In that case, if we consider crack extension by dx, the rectangular region *5678* (Fig. 1d) obtained by moving rectangle *1234* ahead by dx with the crack advance contains the same amount of energy as the original rectangle. The energy contained within rectangle *2673* (Fig. 1d) flows into the rectangle *1234* as it moves forward with the crack, and the energy contained in the rectangle *1584* (Fig. 1d) has flowed out. So the overall energy change is given by the difference between the energies contained within the rectangles *2673* and *1584*, which is what we have calculated.

The energy dissipated per unit length of fracture may be expressed as $W = hG_f$ where G_f is the average (mode I) macroscopic fracture energy of the material over the ice plate thickness; $G_f = K_f^2/E$ where K_f = average fracture toughness [Broek, 1986] and E is Young's modulus of ice. Setting $hG_f = W$ (Eq. 2) leads to the following expression for the critical thermal bending moment at which the fracture must propagate:

$$M_f = \sqrt{\frac{\varrho h G_f}{2\lambda^3}} \tag{2}$$

The difference of ice temperature from the temperature of the sea water, T_0, may be written as $\Delta T(z) = \Delta T_1 f(\zeta)$ where z = vertical coordinate; $\zeta = z/h$ = relative vertical coordinate, ΔT_1 = temperature difference between the top and bottom faces of the plate, and f = function defining the temperature profile (Fig. 1e), which must be calculated in advance. Taking into account that the normal strains in both the x and y-directions as well as the vertical normal stresses are zero, we find that the thermal bending moment in the plate before fracture is $M_T = \int_{-h/2}^{h/2} \hat{E}\alpha\Delta T(z) z dz$, with $\hat{E} = E/(1-\nu)$ where α = coefficient of linear thermal expansion of ice, and the value of the elastic modulus E is taken as the average over the plate thickness. We also ignore the variation of α throughout the plate thickness and take an average value. Substituting for $\Delta T(z)$ and denoting $I_T = \int_{-1/2}^{1/2} f(\zeta)\zeta d\zeta$ (a constant), we obtain $M_T = \hat{E}\alpha\Delta T_1 h^2 I_T$. Fracture will propagate if $M_T = M_f$. From this, the critical temperature difference required for crack propagation is $\Delta T_{cr} = \Delta T_1 = (\hat{E}\alpha I_T h^2)^{-1}(\varrho h G_f/2\lambda^3)^{1/2}$. Substituting now the foregoing expressions for λ and D and rearranging, we obtain the result:

$$\Delta T_{cr} = C_1\, h^{-3/8}, \qquad \text{with} \quad C_1 = \frac{(1-\nu)^{5/8}\,\varrho^{1/8}\,\sqrt{G_f}}{\sqrt{2}\,[3(1-\nu^2)]^{3/8}\,E^{5/8}\,\alpha\, I_T} \tag{3}$$

An important property to note is that the critical temperature difference depends on h, i.e. there is a *size effect* (or scale effect), as shown in the plot of Fig. 2a. According to plastic limit analysis or elastic analysis with allowable stress, there is—as a rule—no size effect, i.e. ΔT_1 is independent of h. The size effect is the salient property of fracture mechanics. In the present problem, the size effect differs from that known for two-dimensional or axisymmetric problems, for which linear elastic fracture mechanics (LEFM) generally predicts the critical stress or critical temperature difference to be proportional to size$^{-1/2}$, provided that geometrically similar structures with similar cracks are considered.

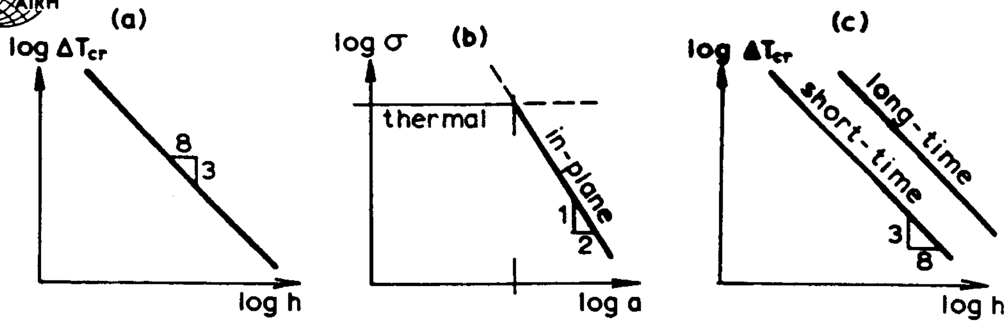

Figure 2: (a) LEFM size effect in thermal bending fracture, (b) transition to in-plane fracture with its size effect, and (c) effect of creep on size effect.

Fracture in Presence of In-Plane Forces

The sea ice plate is normally subjected to a significant in-plane force N, which we consider to be normal to the crack direction (defined as negative when compressive). The second-order geometric effect known from the theory of column buckling must then be taken into account. Therefore, we now generalize the solution to a semi-infinite beam on elastic foundation carrying axial force N. The slope (derivative) w' of the deflection curve $w(y)$ has generally the form (see e.g. Bažant and Cedolin's textbook, 1991, Sec. 5.2) $w' = (A \sin ry + B \cos ry)e^{-sy}$ where $r = \lambda\sqrt{1+\gamma}$, $s = \lambda\sqrt{1-\gamma}$, $\gamma = -N/(2\sqrt{\varrho D})$ and A and B are arbitrary constants. The boundary conditions at the crack, $y = 0$, are $w' = \vartheta$ and $Dw''' - Nw' = -V = 0$ where the primes denote derivatives with respect to y and V = vertical shear force. From these two conditions: $B = -\vartheta$ and $A = -\vartheta[s^2 - r^2 - (N/D)]/2rs$. Substituting this into the expression for the bending moment in the beam, which is expressed as $M = Dw''$ after w'' has been calculated by differentiating w', one gets (for $y = 0$) $M = (1-2\gamma)(1-\gamma)^{-1/2}\lambda D\vartheta$. The energy release rate due to fracture propagation may now be calculated as $W = M\vartheta/2$:

$$W = \frac{\sqrt{1-\gamma}}{1-2\gamma}\,\frac{M^2}{2\lambda D} \tag{4}$$

Setting this equal to the energy dissipated by fracture, $W = hG_f$, and solving for M, one obtains for the critical thermal bending moment M_f causing fracture propagation the result: $M_f^2 = 2h\lambda DG_f(1-2\gamma)/\sqrt{1-\gamma}$. From the condition that $M_f = M_T$, it follows that

$$\Delta T_{cr} = \Delta T_1 = \frac{1-\nu}{E\alpha I_T h^2}\sqrt{2h\lambda DG_f\frac{1-2\gamma}{\sqrt{1-\gamma}}} \tag{5}$$

Finally, introducing the foregoing expressions for λ, D and C_1, and rearranging, one obtains the result:

$$\Delta T_{cr} = C_1\,h^{-3/8}\,(1-2\gamma)^{1/2}(1-\gamma)^{-1/4} = C_1\,h^{-3/8}(1-\frac{3}{4}\,\gamma+\cdots) \tag{6}$$

in which $\gamma = -k_1\overline{\sigma}/\sqrt{h}$, with $\overline{\sigma} = N/h$ and $k_1 = [3(1-\nu^2)/E\varrho]^{1/2}$.

The most interesting aspect property to note is that the critical temperature difference again exhibits size effect. Asymptotically for small $|N|$ the size effect is the same as before.

But as N becomes negative (compressive) of increasing magnitude, the size effect is getting stronger than $h^{-3/8}$, and as N becomes positive (tensile) of increasing magnitude, the size effect is getting weaker. Furthermore, the influence of the average in-plane stress on the size effect diminishes as the plate becomes thicker.

The fact that, for similar temperature profiles, a thicker plate must fracture at a smaller temperature difference can explain Assur's [1963] observation that new fractures in the arctic pack-ice generally do not form along the lines of weakness, such as thin refrozen leads, but run through intact floes and across old pressure ridges. From the strength theory, which gives no size effect, this observation cannot be explained.

If in-plane forces are present, they alone can produce fracture. So they can have a strong effect. In the foregoing we have tacitly assumed that fracture extension causes no release of the energy of the in-plane forces in the ice plate, in the manner of tensile cracks. For the sake of simplicity, assume that $\sigma_{xy} = 0$. The in-plane remote uniform normal stress is $\overline{\sigma} = \overline{\sigma}_{yy} = N/h$. The average stress value that would cause the crack in an infinite ice plate to propagate is [e.g. Broek, 1986]: $\overline{\sigma} = \sqrt{E\overline{G}_f/\pi a}$ in which a is the half-length of the crack (Fig. 1d); $\overline{G}_f$ is the large-scale fracture energy for in-plane through-fracture of the ice plate (which could be much larger than the G_f-value for a crack propagating across the plate thickness), and $\overline{K}_f$ is the associated large-scale in-plane fracture toughness. Note that the strength $\overline{\sigma}$ according to the last expression is independent of the plate thickness.

Because $\overline{\sigma} \propto a^{-1/2}$, the plot of $\log\overline{\sigma}$ vs. $\log a$ (Fig. 2b), is a straight line of slope -1/2. In this plot, the thermal bending fracture is represented by a horizontal line since the crack length a does not appear in Eq. 3 or 6. Obviously, these two straight lines intersect at a certain critical crack length (Fig. 2c):

$$a_{cr} = \frac{E\overline{G}_f}{\pi\overline{\sigma}^2} \tag{7}$$

The value of $\overline{\sigma}$ is basically controlled by the mechanism of ice floe movements in the arctic ocean and is independent of the thermal effects. It follows that, when an in-plane force is present, thernal bending drives the fracture formation and growth only at the beginning. When the crack becomes long enough, namely $a > a_{cr}$, the thermal stresses must cease to matter and the fracture becomes driven by in-plane forces. Thus, steady-state propagation of thermal bending fracture can go on *ad infinitum* only when the in-plane force happens to vanish, $N = 0$, which is an unlikely situation. Otherwise, the thermal bending fracture is only quasi-steady, up to fracture length a_{cr}.

Rate and Time Effects

The foregoing analysis has neglected *creep*, which is very strong in the case of ice. One effect of creep is to relax thermal stresses. This can be approximately taken into account by replacing the value of elastic modulus E in the expression for M_T with the effective (sustained) modulus $E_{eff} = E/(1+\varphi_t)$ where E represents the secant modulus for rapid (nearly instantaneous) loading and φ_t is the creep coefficient, representing the ratio of creep-to-elastic strains for the typical duration t_1 of the temperature difference in the plate. Because of the pronounced nonlinearity of the creep strain as a function of stress σ, the value of this ratio varies through the thickness of the plate as well as along the plate. For the sake of simplicity, we assume that we can approximately take a certain average value of φ_t, determined as the creep-to-elastic strain ratio for a certain average stress level σ_0; then $\varphi_t = \epsilon_t/\epsilon_0$ where ϵ_0 is the instantaneous strain and ϵ_t is the creep strain.

Another effect of creep is to reduce the energy release due to crack propagation. Let t_p be the time that the fracture process zone (the oval-shaped zone F in Fig. 1d) takes to travel across a fixed point. Noting that the value of the cylindrical stiffness D is proportional to E, we can take creep approximately into account by replacing D in the expression for M_f with $D/(1+\varphi_p)$ where φ_p is the creep coefficient representing the value of the creep-to-elastic strain ratio for the time duration t_p. Again, due to nonlinearity of the creep law of ice, this ratio varies through the plate thickness and along the plate but, for the sake of simplicity, is taken as a constant, evaluated for the average stress level σ_0.

One may now retrace the foregoing derivations, replacing E and D as indicated. This shows that Eqs. 3 and 6 remain valid but the expressions for C_1 and k_1 must be modified as follows:

$$C_1 = \frac{(1-\nu)^{5/8}\varrho^{1/8}\sqrt{G_f}}{\sqrt{2}\,[3(1-\nu^2)]^{3/8}\,I_T\,E^{5/8}\alpha}\,\frac{1+\varphi_t}{(1+\varphi_p)^{3/8}}; \qquad k_1 = \sqrt{\frac{3\,(1-\nu^2)}{E\,\varrho}(1+\varphi_p)} \tag{8}$$

We see that the creep prior to fracture, which relaxes the thermal stresses, tends to increase ΔT_{cr} (Fig. 2c), while the creep during fracture, which reduces the energy release rate, tends to decrease ΔT_{cr}. For rapid fracture, the latter effect may be neglected ($\varphi^p = 0$).

Numerical Example

Consider the following typical values of material parameters: $\varrho = 9810$ N/m^3, $\nu = 0.29$, $\alpha = 5 \cdot 10^{-5}/°C$ [Weeks and Assur, 1967, Butkovich, 1957]. The value of the instantaneous (dynamic) elastic modulus E_0 is $E_0 \approx 7$ GPa. For our simplified analysis, however, we need to include the primary (short-time) creep into the apparent elastic (short-time) deformation, which means that we need to use for E the apparent elastic modulus value obtained in conventional static tests in laboratory testing machines, approximately $E = 1$ GPa (as used by Evans, 1971). According to Urabe and Yoshitake [1981], Weeks and Mellor [1984] and Sanderson [1988, p.91], we may use $K_f = 0.1$ MN $\cdot$ m$^{-3/2}$, and the corresponding fracture energy value of sea ice is $G_f = K_f^2/E = 10$ N/m.

With respect to creep, we need to estimate at least roughly the average magnitude of thermal stresses. If the secondary creep is taken into account using Norton's law, the stress-strain relation may be approximately written as $\dot\epsilon = E^{-1}\dot\sigma + k_c\sigma^3$ where t = time, the superior dots denote time derivatives and $k_c = Ae^{-Q/RT}(1-\sqrt{v_b/v_0})^{-3}$ [Sanderson, 1988, p. 82], in which Q = activation energy of creep, R = gas constant, $Q/R = 7818$ °K, $A = 3.5 \cdot 10^6$ (MPa)$^{-3}$ s^{-1}, $v_0 = 0.16$, and v_b is the porosity due to brine pockets, which we take as $v_b = 0.06$. Considering temperature -40°C, i.e. $T = 233$ °K, we get $k_c = 161 \cdot 10^{-9}$ (MPa)$^{-3}$ s^{-1}.

Suppose now that a dramatic temperature drop of $\Delta T = 40$ °K occurs over the period of

Δt = 14 days. According to the effective modulus method, the creep value may be approximately based on the final stress σ. To get its crude estimate, we may write for uncracked ice at the top plate surface an incremental uniaxial stress-strain relation $E^{-1}\sigma + k_c\sigma^3\Delta t = \alpha\Delta T$. Substituting the aforementioned values and solving this cubic equation for σ, we get the estimate σ = 0.209 MPa. The basic equation of the effective modulus method is $\epsilon = \sigma/E_{eff}$ with $E_{eff} = E/(1+\varphi_t)$ and $\epsilon = \alpha\Delta T$. From this we solve $\varphi_t = (E\alpha\Delta T/\sigma) - 1 = 8.57 \approx 9$. To estimate the value of I_T, we assume the characteristic temperature profile to be a cubic parabola with a zero slope at the bottom of the plate, for which one obtains $I_T = 3/40$. The following estimates then ensue:

$$\text{For } h\text{=1m: } \Delta T_{cr} = 24.6°\text{K}; \quad \text{for } h\text{=3m: } \Delta T_{cr} = 16.3°\text{K}; \quad \text{for } h\text{=6m: } \Delta T_{cr} = 12.6°\text{K}. \quad (9)$$

These values are often exceeded by the arctic weather. This means that the thermal bending moments are indeed capable of causing a bending fracture through the whole thickness of the ice plate (but other possible fracture mechanisms, of course, are not excluded by this result).

It must be emphasized that this result is applicable to the case of approximately similar temperature profiles. If h is increased, the time to reach a similar profile increases proportionally to h^2, but the surface temperature change may not be sustained long enough or the surface temperature history may become more complex. Thus, in practice, the profiles are unlikely to ever become exactly similar. One important point, however, should be noted: The most critical temperature profiles, which maximize M_T according to the elastic calculation we used, are similar. This is the main practical justification for the applicability of the assumption of similar profiles.

SIZE EFFECT IN VERTICAL PENETRATION THROUGH ICE PLATE

Consider now another important problem: the maximum load P required for the penetration of an object through the arctic sea ice plate, either from top (the problem of bearing capacity) or from below. This problem has been studied extensively on the basis of either elasticity theory with an allowable stress limit or plastic limit analysis [Kerr 1976, 1991]. Fracture mechanics, which is much more realistic for sea ice, apparently has not yet been applied. We will attempt it now, considering a small punch such that load P is applied as an almost concentrated force. From observations it is known that the failure process involves the propagation of n radial cracks (Fig. 3a, typically $n \approx 4$ to 12), followed by the formation of circumferential (or "hoop") cracks (Fig. 3b) at radial distance a_m.

Approximation by Narrow Wedge Beams

For the sake of simplicity, we now assume that the cracks are through-cracks all the way to their tip. This is certainly an idealization, but is necessary to permit the use of plate bending theory in conjunction with LEFM. Further we assume that the cracks relieve all

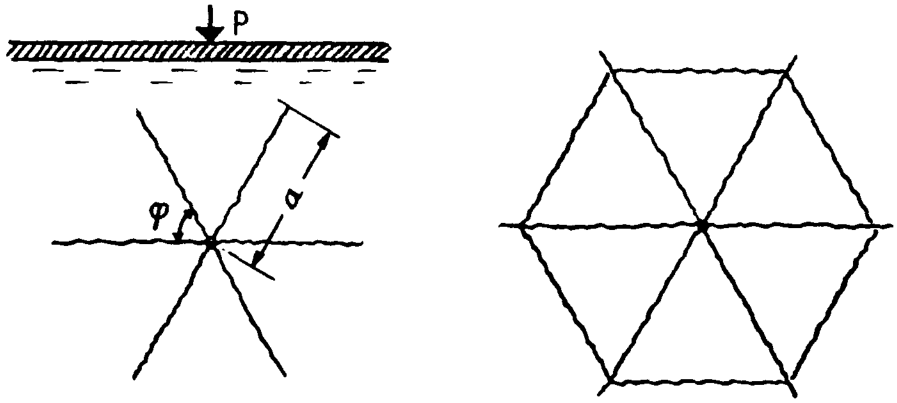

Figure 3: Typical cracking pattern caused by vertical penetration; (a) radial cracks and (b) subsequent circumferential cracks.

the bending energy of the circumferential bending moments M_ϑ everywhere up to the radial distance a where a = radial crack length. This assumption means that, for $x \leq a$ (where x = radial coordinate), the plate is replaced by infinitely many continuously distributed infinitely narrow radial wedge beams resting on an elastic foundation. The wedge beams have a variable width $xd\vartheta$ and transmit bending moment (per width $xd\vartheta$) $\overline{M}d\vartheta = xDw''d\vartheta$ and shear force $\overline{V}d\vartheta = (d\overline{M}/dx)d\vartheta$ (this assumption overestimates the energy release due to fracture, and so it may be expected to underestimate the failure load). For $x > a$, we have axisymmetric bending of an infinite plate on elastic foundation. The problem is, therefore, one-dimensional. The governing differential equations are:

$$\text{for } x \leq a: \qquad \frac{1}{x}\frac{d^2}{dx^2}\left(x\frac{d^2w}{dx^2}\right) + \frac{q}{D}w = 0 \tag{10}$$

$$\text{for } x > a: \qquad \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)\left(\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dx}\right) + \frac{q}{D}w = 0 \tag{11}$$

At the interface, $x = a$, w and w' must be continuous. The boundary conditions are: for $x = 0 : P = -2\pi D(xw'')' = -2\pi Dw''$($M = \overline{M} = 0$ is automatic), and for $x \to \infty : w = w' = 0$. Let w_0 be the value of w at $x = 0$. (The solution of the differential equation (10) alone, for $w = w' = 0$ at $x \to \infty$, was given by Nevel [1958, 1961] in the form of an infinite series.)

Introduce now non-dimensional variables $\xi = x/L$ and $\zeta = w/L$ where $L = (D/\varrho)^{1/4} = (\lambda\sqrt{2})^{-1}$ = decay length (i.e., characteristic length for the decay of flexural disturbances along the plate), and denote $\overline{a} = a/L$. The foregoing two differential equations then take the form:

$$\text{for } \xi \leq \overline{a}: \qquad \frac{1}{\xi}\frac{d^2}{d\xi^2}\left(\xi\frac{d^2\zeta}{d\xi^2}\right) + \zeta = 0 \tag{12}$$

$$\text{for } \xi > \overline{a}: \qquad \left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi}\right)\left(\frac{d^2\zeta}{d\xi^2} + \frac{1}{\xi}\frac{d\zeta}{d\xi}\right) + \zeta = 0 \tag{13}$$

At $\xi = \bar{a}$, both ζ and ζ' must be continuous. The boundary condition at $\xi = 0$ may be considered as $\zeta = w_0/L$ (where w_0 = load-point deflection), and for $\xi \to \infty : \zeta = \zeta' = 0$. Introduction of the nondimensional variables achieved that the physical constants disappeared from the differential equations; a physical constant, namely $\bar{a}$, is present only in the interface condition, aside from w_0 in the boundary condition. Because of linearity of the problem, the solution ζ is proportional to w_0.

Let us now define $\zeta = F(\xi, \bar{a})$ as the solution of the foregoing two differential equations for the boundary condition $\zeta = 1$ at $\xi = 0$, the other boundary and interface conditions being the same. Then

$$w = w_0 F(\zeta, \bar{a}); \qquad P = 2\pi w_0 \sqrt{\varrho D}/f_1(\bar{a}) \tag{14}$$

where $1/f_1(\bar{a}) = [d^2F(\xi,\bar{a})/d\xi^2]_{\xi=0}$. Calculating now the load-point compliance $C(\bar{a}) = w_0/P$, applying the well-known fracture mechanics formula $[\partial\Pi/\partial a]_{P=const.} = (P^2/2)dC(\bar{a})/da$ (= energy release rate, and Π = complementary energy of the system), and imposing the energy balance condition for crack propagation, which reads $\partial\Pi/\partial a = nhG_f$ where $n = 2\pi/\varphi$ = number of radial cracks and φ = angle between the adjacent radial cracks (Fig. 3), one finds that the radial cracks are critical (i.e. can propagate) if $P = 2\sqrt{\pi nhG_f}(\varrho D^3)^{1/8}/\sqrt{f_1'(\bar{a})}$. Expressing D in terms of h, we thus obtain for the applied nominal stress $\sigma_N = P/h^2$ and the corresponding load-point deflection during radial crack propagation (at no hoop cracks) the following results:

$$\sigma_N = C_P[f_1'(\bar{a})]^{-1/2}h^{-3/8}; \quad \text{with } C_P = (4/27)^{1/8}\sqrt{\pi n G_f}\varrho^{1/8}[E/(1-\nu^2)]^{3/8} \tag{15}$$

$$w_0 = C_w f_1(\bar{a})[f_1'(\bar{a})]^{-1/2}h^{1/8}; \quad \text{with } C_w = (\sqrt{3}/\pi)C_P[(1-\nu^2)/\varrho E]^{1/2} \tag{16}$$

According to this result, there is again a size effect on the nominal strength. The factor $h^{-3/8}$ is the same as before, however the $(-3/8)$-power size effect is modified since the value $\bar{a} = \bar{a}_m$ of the relative crack length $\bar{a} = a/L$ at maximum load must depend on h. To clarify this dependence, consider the role of the circumferential cracks. The load-deflection curve $P(w_0)$ during the growth of radial cracks (at no circumferential cracks) must no doubt be monotonically rising, with no peak. We will assume the circumferential cracks to grow simultaneously across the plate thickness (although in reality they could also propagate along the plate, which we neglect). Since the full formation of the circumferential cracks creates a mechanism and thus reduces P to zero, one may expect that, during the growth of the circumferential cracks across the plate thickness, the load P must be decreasing. By this argument, the peak load P should occur right at the start of growth of the circumferential cracks from the surface, which is governed by the tensile strength σ_t of ice. So we need to calculate the surface bending stress, which is $\sigma = 6M/h^2 = (6D/h^2)w'' = (6Dw_0/h^2L^2)d^2F(\xi,\bar{a})/d\xi^2$. Setting this equal to σ_t, and imposing the necessary condition of bending stress maximum ($d\sigma/d\xi = 0$), we get:

$$f_1'(\bar{a}) = \left(\frac{3C_P}{\pi\sigma_t} f_1(\bar{a}) \frac{d^2F(\xi,\bar{a})}{d\xi^2}\right)^2 h^{-3/8}; \qquad \frac{d^3F(\xi,\bar{a})}{d\xi^3} = 0 \tag{17}$$

This represents a system of two nonlinear algebraic equations, whose solution may be denoted as $\xi = \xi_m(h)$ and $\bar{a} = \bar{a}_m(h)$. or a given h, these functions give the coordinate of the maximum stress point and the radial crack length for which the maximum bending stress becomes equal to the strength.

The variation of $\bar{a}_m$ with increasing h cannot be strong since the point $\bar{a}_m$ must lie within the first half-wave of the deflection curve. Moreover, the first equation (17) indicates that $f_1'(\bar{a}) \to 0$ for $h \to \infty$. Thus h disappears from equations (17) for large thicknesses, which means that ξ_m and $\bar{a}_m$ become independent of h. Consequently, the $(-3/8)$-power law is asymptotically approached for large thickness h. To decide whether this power law is approximately applicable for realistic ice thicknesses will have to await a numerical solution.

General Size Effect in Floating Plate under Concentrated Load

A more realistic solution would require two-dimensional analysis of the energy release from a wedge-shaped floating plate of a finite angle ϕ. This could, of course, be done by finite elements. The type of size effect, however, cam be deduced without actually obtaining the solution.

The deflection surface $w(x,y)$ of a floating plate under distributed load $p(x,y)$ is governed by the differential equation:

$$D\nabla^4 w + \varrho w = p(x,y) \tag{18}$$

Consider now concentrated load P at point $x = y = 0$. Then $p(x,y) = P\delta(x,y)$ where $\delta(x,y)$ = two-dimensional Dirac delta function, such that $\int\int \delta(x,y)dxdy = 1$. Introducing nondimensional variables $\xi = x/L, \eta = y/L$ and $\zeta = w/L$, and noting that $p = \delta(x,y)P/L^2$, Eq. (18) is transformed to:

$$\nabla^4\zeta + \zeta = \delta(x,y)(PL/D) \tag{19}$$

Consider now that the plate is infinite, with the boundary condition $w = 0$ at infinities. Let $\zeta = F(\xi,\eta;\bar{a})$ be the solution of the differential equation:

$$\nabla^4\zeta + \zeta = \delta(x,y) \tag{20}$$

for a plate containing a crack of relative length $\bar{a} = a/L$ (with the proper boundary conditions written for the crack surfaces). Then the solution of differential equation (19) is $\zeta = F(\xi,\eta;\bar{a})(PL/D)$. Now the complementary energy of the floating plate may be calculated as $\Pi = Pw_0/2 = PL\zeta_0/2 = F_0(\bar{a})p^2L^2/(2D)$ where we denoted $F_0(\bar{a}) = F(0,0;\bar{a})$. For the energy release rate we now have the condition $[\partial\Pi/\partial a]_{P=const.} = [\partial\Pi/\partial\bar{a}]_P/L = F_0'(\bar{a})p^2L/(2D) = hG_f$. Solving P from this equation, calculating the nominal strength $\sigma_N = P/h^2$ and expressing L and D in terms of h, we get the result:

$$\sigma_N = C_0 h^{-3/8}; \qquad \text{with} \qquad C_0 = \left(\frac{\varrho}{108}\right)^{1/8}\left(\frac{E}{1-\nu^2}\right)^{3/8}\sqrt{\frac{G_f}{F_0'(\bar{a})}} \tag{21}$$

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