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ANALYSIS OF FRAMED STRUCTURES

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PART I

Framed structures are systems of bars with rigid joints. The rigidity of joints causes the bending of bars with equal rotation of all bars connected in a joint. Trussed girders, with (actual or supposed) hinged connections at the joints are determinate or overdeterminate in form, whereas framed structures (without diagonals) having hinged joints are geometrically indeterminate. With rigid joints, the bars in *plane systems* are acted on by axial forces, bending moments, and shearing forces. This greatly increases the number of unknowns, framed structures being many times statically indeterminate. Exact solution with usual methods is complicated and laborious. The greatest possible simplification of analysis for practical purposes is the aim of all new publications.

As in the case of all hyperstatic systems, framed structures can be analyzed by Catigliano's *method of least work* (1). The statically indeterminate quantities are the components of reactions or internal forces (axial and shearing forces, bending moments) in chosen sections. Expressing by these unknowns the components of internal forces in all sections of the bars, we derive by the theorem of least work as many equations as there are unknowns. We then have a great number of linear equations. This method of analysis is simple if each equation contains only a small number of unknowns. For this purpose, it is possible to choose the sections containing hyperstatic components and their directions so that each hyperstatic quantity produces deformation in the least number of directions, least affects the other unknowns (2,3). For a closed or fixed-end simple frame, each elastic equation can contain only one unknown if the components of internal forces in a section or of reactions are transferred to the elastic center of the frame.

The analysis of continuous beams with unyielding supports successfully uses *fixed points*, introduced by C. Culmann (4). This method of analysis can be applied also to framed structures (5). If only one member in the system is loaded, the bending moments in each unloaded member are given by a straight line which meets the axis of the member in a fixed point. Calculating in advance the fixed points in all members, we can determine the bending moments in all members for only one member loaded; from bending moments, shearing and axial forces. This method is advantageous for a frame whose joints are immovable, as is the case for a symmetrical and symmetrically loaded frame or for a frame whose supports prevent displacement of joints for every loading. If the joints move, equilibrium cannot be calculated as in the case of immovable joints. It is then necessary to put the system in equilibrium by suppressing the forces obtained in some joints as the resultants of internal forces; a new calculation must be made.

Instead of stress or reaction components, the analysis of frames can use as unknowns components of deformation; e.g., joint and bar rotations. The number of these unknowns is generally much smaller as compared to the number of statically indeterminate components of internal or external forces; this substantially facilitates the calculations. The origin of this slope-deflection method can be traced to J. Cl. Maxwell (6). For the solution of secondary stresses in trusses, this

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method was applied by H. Manderla (7), E. Winkler (8) and O. Mohr (9). For frames, this method was systematically elaborated by A. Ostenfeld (10). Slope-deflection method can be applied for simultaneous loading of many or all members.

The basic equations for calculation by the slope-deflection method are joint equations (following from equilibrium of bending moments in joints) and bent equations (expressing the equilibrium of horizontal components of external and internal forces for a section through one story). With respect to unknown angles, all these equations are linear equations with a large number of unknowns. Precise solution is possible by successive substitution, or by Gauss' method of elimination, for a determinant symmetrical to the diagonal by shortened elimination (11). A great number of equations can be solved by successive approximation or iteration. For frames with many stories and bays, we can choose a simpler system as a part of the given structure and calculate from it the original system.

The laborious solution of the slope-deflection equations can be avoided by the *method of moment distribution,* which calculates successively, and very simply, the end moments in members. The method is due to H. Cross (12,13). Similar to the method of fixed points, this method calculates directly the distribution of resultant (unbalanced) moments at the joints in the members meeting at a joint, as well as their carry-over to the other end of each member. After determining the distribution and carry-over factors, end moments in members can be calculated directly by successive approximation, which allows an exactness to any desired degree. The method has the advantage that errors of calculation can be improved by further distribution without correcting the previous solution. This method is very simple in principle. It solves no equations, and all computations consist of the simplest arithmetic. Especially simple is the analysis of frames whose joints do not move. But also for moving joints, this method is advantageous if moment distribution is combined with the distribution of forces acting in horizontal sections through stories (14). Another advantage is that bending moments necessary for proportioning of sections can be derived directly.

Another method of frame analysis uses deformation of the system, determined by joint rotations, as unknown quantities. This method first calculates primary joint rotations, caused by the loading of members connected at the joint, and distributes them to next joints as secondary rotations which add to the primary. Repeating this solution, any desired accuracy can be achieved. This method, called *distribution of deformation,* was first applied by G. A. Maney and W. M. Wilson (15) and was improved in later publications (e.g., 16). The late C. Klouček occupied himself in detail with this method (17,18). In many cases the method gives precise results at once, without repeated calculations.

Derived originally for the calculation of hyperstatic pin-joined trusses, the *relaxation method* was also used by its author, R. V. Southwell (19), for analysis of continuous beams and frames; for continuous beams, the method is identical with the older moment distribution method. The relaxation method can be applied analogically to other physical problems (electrical networks, vibrations, elastic stability, stiffened suspension bridges, etc.).

The foregoing methods are good for any frame system. There are also methods for special cases. One of them is the *method of four moment equations* (20), using bending moments as hyperstatic quantities. This method solves simpler systems: continuous frames of one story and one-bay multistory frames.

Successive approximation methods also include the *panel method,* starting from a quadrilateral panel as an element of the structure (21). It has been applied to one-bay frames with several stories, having vertical or inclined columns; also for the Vierendeel girder (22) and for the complicated case of a hingeless spandrel-braced arch (23).

In simple cases (continuous beams and frames, closed frames) a quick calculation is given by the *method of relative flexure factors* (24,25). This method determines very simply the tangents to the elastic line in joints of the system loaded in any member with bending moment at the joint. Thus is derived the elastic line of the system and the influence line of end-moment in a member.

All the foregoing methods concern plane frames. There is an abundant literature dealing with them in many languages (Czech, Italian, Polish, Russian, etc). Their theory is very advanced, as concerns the numerous methods and their practical applicability. The analysis of *space frames* is much more complicated. It is necessary to consider in every joint components of internal forces in three perpendicular directions, and also moments (bending or torsional) relative to all three axes. The number of components of deformation increases in the same proportion. The literature about space frames is comparatively rare. See e.g., Richards (26) and Bažant (27).

PART II

In the eleven years since Part I was written, the frame analysis has been going through a rapid development and has reached remarkable achievements in general formulation as well as in solutions of new practical problems. The impetus behind this development has been the broad introduction of *electronic computers* which have enormously increased the speed of computations as well as the possibilities of solving more complex and intricate structures. On the other hand, the use of computers required the creation of more general and common formulations. The latter aim has been achieved by the introduction of ***matrix algebra*** in the analysis (special survey, see Argyris (28)) which allows an excellent systemization and simplification of calculations and, at the same time exceptional conciseness and transparency of mathematical procedure.

Owing to the general character of solutions the term of framed structures has been usually taken in a more general way. It is defined as a complex structure—a finite assembly of mutually connected one-dimensional members, i.e., bars, which are subjected not only to normal forces but also to bending or torsional moments. In this light, the notion of framed structures includes not only the rigidly jointed frames in their classical sense, but also frames with pin-joints, semi-rigid joints and other connections, arches, grid frameworks, etc., all of which can be analyzed in general by the same methods.

The use of electronic computers with the help of matrix algebra allows the solution of many statically indeterminate structures leading to a very large, but not unlimited, system of equations. However, the benefit from electronic computers would be small if their use were limited only to the solution of these equations. The determination of their coefficients from initial data, characterizing the structure as well as the derivation of all internal forces and displacements from solved unknowns, represents an oft laborious task and therefore it is necessary to express all calculations in matrix algebra.

In *matrix analysis of frames* the parallel to all methods described in Part I may be given. However, the importance of the two *basic methods*—the *force method* and the *displacement method*—has been increasing, while the help of electronic computers enables the solution of considerably larger systems of equations. The force method is also called the method of least work (Part I), flexibility (influence) coefficient method, flexibility matrix method, action method or compatibility method. The displacement method represents a generalization of the slope-deflection method (Part I) and is called also the equilibrium method or stiffness (coefficient) matrix method. The principle of the force [displacement] method is in using internal forces [displacements] as unknowns. Most generally they were presented in Argyris' articles (29,30). Let us note that a generalized mixed method can also be introduced, in which simultaneously internal forces and displacements are taken as unknowns (28,31).

The force and displacement methods represent two ***dually corresponding techniques.*** Namely, they are formally identified as forces (and moments) in one method are replaced by the displacements (rotations, deflections) in the other method. Simultaneously, the compatibility (kinematic) conditions correspond to the equilibrium conditions and so forth.

Let us demonstrate the ***procedure*** of both methods (dual terms are written in brackets). The solution by the force [displacement] method is based on two rectangular matrices. The elements of the first one are the internal forces [displacements] which statically [kinematically] correspond to given external loads [prescribed displacements] in the chosen *primary* [or basic] static [kinematic] *system*, i.e., in the frame in which the statically [kinematically] indeterminate quantities, called also ***redundants,*** are taken as zero. The elements of the second one are the internal forces [displacements] which statically [kinematically] correspond to unit values of individual statically [kinematically] indeterminate quantities in the primary system without external loads and prescribed displacements. With the help of the second matrix, the diagonal *flexibility* [*stiffness*] *matrix* for the chosen primary static [kinematic] system is derived. Using further the first matrix and the column matrix of prescribed external loads [displacements], the column matrix of load terms is found. The matrix equation for the unknown statically [kinematically] indeterminate quantities follows then from the compatibility [equilibrium] conditions applied to the chosen primary system. It is simply obtained that the column matrix of load terms is equal to the product of the diagonal flexibility [stiffness] matrix of the primary system and the column matrix of unknown statically [kinematically] indeterminate quantities. Their solution is then given by *inversion* of the flexibility [stiffness] matrix of the primary system. All the internal forces [displacements] are obtained by multiplying the column matrix of indeterminate quantities with the second rectangular matrix, mentioned above, and adding the first rectangular matrix. It is clear that all needed matrix operations are the elementary one.

The equations for statically [kinematically] indeterminate quantities may be also derived from dual *energy theorems* (e.g., Argyris (29), and Matheson (36)), which select from all statically [kinematically] possible states of stress [strain] of the frame this one which fulfils the compatibility [equilibrium] conditions. For this purpose may be used the second Castigliano theorem for minimum of the *complementary potential energy* which is the consequence of the principle of *complementary virtual work* (or of virtual forces) [the first Castigliano theorem for minimum of the *potential energy*, which is the consequence of the principle of *virtual work* (or virtual displacements)]. In the case of linearly elastic material the work and energy is equal to the complementary one. Different energy quantities are here introduced in order to give validity to the corresponding theorems in a general case of physically nonlinear material.

The determination of the *flexibility* [*stiffness*] *coefficients* and of load terms may be done, most generally, with the help of the principle of complementary virtual works [of virtual works] (unit load method, Mohr's dummy load method [unit displacement method] (29,30,34)). Here, it should be noted that if the primary statical system is chosen as statically indeterminate, the virtual forces, integrated in product with the deformations of the redundant primary system in order to obtain its displacements (flexibilities, load terms), can be taken on a statically determinate primary system. This simplifies the calculation. This statement (e.g., Beyer (32), Dašek (33)), denoted as reduction theorem, follows directly from the general form of the principle of complementary virtual work (29).

In matrix solution there is often used the direct determination of flexibilities and stiffnesses by *superposition and transformation of flexibilities or stiffnesses* of single members and parts (along certain series of bars) according to relations corresponding to the transformation of coordinate systems (e.g., Hall (34), Shore (35)). The stiffness matrix of a member is there the inverse of its flexibility matrix, but only when the column matrices of statically and kinematically indeterminate quantities mutually correspond.

If bar elongations can be neglected, it is also possible to determine the flexibilities and load terms in a similar way as in Mohr's well-known theorem of conjugate beam (36,37). This theorem is used in the moment-area method (37,38), or method of elastic weights (loads) or in string polygon analysis (39), which are not usually presented as a special derivation of force method equation, in spite of being so. Further extent of this method is the method of fictional equilibrium conditions (40) and the *conjugate frame method* (41). General form of this method was developed by the author in (42) where there was derived the systematic way of determining the conjugate frame to any frame by replacing all supports and internal connections of bars by their conjugates. This method is based on the force-deformation analogy, i.e., on analogy of vectorial equation for the course of internal forces in the bar, resulting from equilibrium, and of vectorial equation of deflection line of bar, resulting from compatibility. It determines the displacement vector as an internal force vector on the conjugate frame produced by conjugate loads which are equal to vectors of curvature change of the actual frame caused by bending. Thus, any deformation of the frame, e.g., flexibility of the primary system, can be obtained with the help of equilibrium conditions. Furthermore, the compatibility conditions of the actual n-times statically indeterminate frame can also be replaced by the equilibrium conditions of the n-times kinematically indeterminate conjugate frame. This procedure is advantageous for frames with one (or two) story or bay and is also suitable for transversely curved beams.

Besides physical characteristics of material and geometric data about spatial position and dimensions of each member, the information about interconnection of members, having in principle a topological character, are necessary for the derivation of flexibility or stiffness matrix and of load terms. For an intricately interconnected frame the *use of topology* (43) may be therefore advantageous and it can give a more general approach to solution procedure (e.g. (44–49)). Interest about this subject in recent years has grown and remains high. Using the analogy with topological network theory, the force and displacement methods were presented as counterparts of mesh and node-methods (49) for network analysis, in which the interconnection of members is expressed by branch-mesh and branch-node matrices. The use of the linear graph theory for rigidly jointed frames and of the system theory was made in (46). Also, statical indeterminacy and stable form of structure was successfully studied with the help of these methods (48). In application to the structural analysis, Kron derived for networks the technique of piecewise analysis by tearing and interconnecting the structural parts (44), which, when used for frames (47), is equivalent to the solution with the help of the statically indeterminate primary system.

From force or displacement methods one should be chosen, of course, which leads to smaller number of unknowns in the given case. Then the simplicity of calculation depends on intelligent *selection of redundants*. As was mentioned in Part I, the solution becomes simpler if the redundants are chosen in such a way that the greatest possible number of nondiagonal flexibility (stiffness) coefficients is equal to zero. This condition can be generally achieved by *orthogonalization of the set of redundants* (e.g., Kohl (50) which means that the virtual stress distribution and the strain distribution, corresponding in the primary system to the redundants, are orthogonal functions. A special case of orthogonalization is represented by introduction of the elastic center for simple frame or arch (Part I) and by its further extension to Cross' column analogy (34,36,37) which also includes the prescription for superposition of influences of redundants in order to obtain any internal force. In general, for any frame a partial or total orthogonalization of redundant quantities can be achieved by forming certain independent linear combinations of redundants, which means that new *group redundants*, i.e., groups (combinations) of primarily chosen redundants, ought to be taken into account instead of original single redundants. Evidently, then, the redundants need not be represented as forces in single cuts of bars. They can represent an arbitrary linearly independent self-equilibrating stress system in frame (12,32,34,50,51); this idea descends from Müller-Breslau, Mohr and Beyer (52).

Furthermore, of course, a great simplification is obtained for *symmetric* frames if the given loading is decomposed to the symmetric and antimetric part and if a primary system with symmetric pairs of forces or displacements as group redundants is chosen (32,50,51).

The number of unknowns is lessened and the calculation is often shortened if the *primary* (basic) *system* is chosen as *statically indeterminate* (32,33, 37). Formally this is identical to the inversion of the flexibility matrix by partitioning (34,36), but the mechanical interpretation of this procedure can help in choosing a suitable mode of partitioning.

In the *displacement methods* various unknown and corresponding systems of *basic elements* are introduced. In contradistinction to the slope-deflection method, with respect to simplicity of programming, for matrix analysis it is better to use as unknowns, more generally, the joint displacements (51,53) instead of bar rotations, although the number of unknowns increases. Then, instead of bent equations for stories the equations of shear and normal forces equilibrium in each joint need to be used. This is necessary, e.g., if axial bar elongations are considered or if the bars have curved or broken axis (37). Bars as basic elements can also be used for grid frameworks (54).

The number of unknowns is reduced if a system of more *complex basic elements* is introduced, which is the dual technique to the choice of the statically indeterminate primary system and is equivalent to the inversion of stiffness matrix by partitioning. Thus, for instance, for a large multi-story and multi-bay frame, unknown joint rotations can be selected in each second joint in a chessboard order if crosses of four bars are taken as basic elements (55).

For continuous frames with hinges at midspans it is advantageous to take the deflections of hinges as unknowns, which means that T-shaped basic elements (56) are considered. Here, it should be noted that in the latter case as well as for the solution by force method (57) with shear forces in hinges as redundants, a system of equations, each containing only three unknowns, is obtained. Hence, solutions of these frames can be built up in an analogous manner as the solutions for continuous beams.

Matrices of three-term (or five-term, seven-term) equations, called also *band matrices*, appear for various simple types of frames and their inversion is easier (58,59).

The inversion of matrices for *regular frames* with repeated identical parts, e.g., with members of equal length and stiffness, can also be done in a simplified manner (60). Namely, the analysis leads then to a system of equations, for instance, of slope-deflection equations, in which certain groups of equations have identical form and equal coefficients at unknowns and can therefore be written as *difference equations* with a variable index of unknowns (60a, 60b). The solution of linear algebraic difference equations with constant coefficients (or of systems of such equations) is found as a sum of a particular solution and of linear combinations of powers of roots of corresponding characteristic equations. The general constants are determined from boundary conditions. This method can be used, e.g., for regular continuous frames or tall building frames in which certain approximate irregularities can be allowed (60a), for regular circular space frames (60b), etc.

Analogically to the fixed point method (Part I) in matrix form the *method of coefficients of restraint* can be built up (34). In principle, it is based on the force method and corresponds to the step-by-step inversion by partitioning of the flexibility matrix for a primary system with moments at joints as unknowns. Let us note that there could be derived a dual matrix technique, analogic to the method of fixed points for joint rotations (50), which is based in principle on the slope-deflection method and needs no distribution factors in joints.

The programming is easier for a frame of uniform and regular type. For a frame which is irregular because of various cut-outs and missing members but which can be made regular and soluble by standard programs after adding some further members, the *technique of cut-outs and modifications* ((29), no. 3) can be used advantageously. Firstly, the stresses in the regular frame with added members are solved and then are added the stresses which are produced in added members by such initial strains which make the resulting stresses in added members equal to zero.

Besides the force and displacement methods and other related methods there is possible an other approach, called the *method of transfer matrices*, or in German *Reduktionsmethode* (51). It was, in principle, already used in works of Krylov, Macauley and others (61,62), but it won greater importance after being developed for frames in matrix form by Falk (63). This method is suitable for continuous beams, but it is also usable for simpler frames with open form (continuous frames) or eventually with simple closed form. Whereas the force [displacement] method corresponds in reality to the integration of the fourth order differential equation of bending (torsion, tension) of bars, at which only forces [displacements] in different points of bars are taken as integration constants, the method of transfer matrices starts form this differential equation, taking as integration constants all internal forces and displacements, which are written as a row matrix, in one point, namely, at the end of the bar. The corresponding row matrix for the other end of the bar is then expressed by multiplication with the rectangular transfer matrix, which includes the stiffnesses and loading of bar. By chain multiplication of single transfer matrices for continuous series of bars (and of matrices for transformation of coordinates according to the angle between bars at joints and of matrices expressing the supports conditions) the relation between column matrices at the ends of this series is obtained, yielding equations in unknown forces and displacements. If the series of bars is long, the procedure is sensitive to numerical accuracy. A similar procedure is represented by the matrix progression method (63) or by the method of initial parameters used for combined simple and bending torsion of continuous beams (64). In principle, this method is also related to the traverse method (25).

The *iterative methods*, which determine the solution by successive approximation, still keep importance and, adapted to matrix form (67), allow the solving of considerably large frames with the help of small-size computers. In addition to the moment (force) distribution, to the distribution of deformations and to the relaxation method, which were reviewed in Part I, a new iterative method was presented by Kani (65). This method was later adapted to matrix form (66,67). *Kani's method* is relatively simpler and gives better convergence, if the multistory frame is subjected to sideways. As well as the methods of distribution of moments or of deformations, this method is also based on slope-deflection equations and uses bending moments as iterated quantities. Whereas in each step of moment distribution the unbalanced moment, resulting from equilibrium condition of the joint, is distributed to bars meeting in the joint and then is carried over these bars to their opposite ends, in Kani's method, in each step, the moments acting on the joint are calculated from the opposite end moments according to the equilibrium condition of this joint. In the case of sideways these steps are alternated with similar steps according to the equilibrium of the entire story. Kani's method does not operate with total moments, as in moment distribution, but during all iteration the bending moments are divided into four parts. The first part is the constant fixed end moment. Two further parts concern the rotation of the corresponding joint and of the opposite end joint. The last iterated part expresses the relative displacements of the ends of the bar and determines simultaneously the shear force. Thus, this procedure is, indeed, very close to the distribution of deformations (Part I).

Described iterative methods can be conceived in an almost general way. For instance, the moment distribution method was adapted for space frames (35,51), arch frames (68), grid frameworks (69,70) and others.

There were presented also various modifications of these methods. To speed up the moment distribution method for sideways at multistory frames suitable similar techniques were presented by

Csonka (71,74) Zaytseff (72), Oswald (see also 73,75), Naylor, Bolton (36) and others (61). For instance, according to the approximate representation by simpler *substitute frame* or even by column, the step of joint rotation is done simultaneously with horizontal displacement.

Improving of convergence in relaxation method is achieved by *group relaxation* steps (19,34).

For large frames Krylov (55) reduces the number of unknowns as well as speeds up the convergence of moment distribution (because of smaller carryover factors) by introducing redundant restraints in each second joint in a chessboard order, at which there need be considered, however, more complicated complex basic elements, crosses of four bars instead of single bars. The latter technique corresponds to the selection of a (kinematically) redundant primary system, i.e., assembly of the crosses, at which is iterated the entire system.

Somewhat similar, but a contrary technique of iterative methods which corresponds to iterative solution of a redundant primary system and speeds up the convergence could be derived, if during the iteration certain load terms or certain redundants are taken as arbitrarily variable and if they are changed after each iteration cycle in such a way that the residues, e.g., unbalanced moments or forces in redundant locks (noncompatible part of displacement in redundant cuts) will be minimized, i.e., approximately the sum of the squares. This would be especially effective as a modification of the relaxation method, derived in (76), at which the residues are simply transferred to the chosen arbitrarily variable load terms or redundants. The solution for prescribed values of n load terms which have been chosen arbitrarily as variable or the solution accomplishing the compatibility in cuts of n chosen variable redundants would then be obtained by linear combination according to n different independent resulting states of chosen arbitrary variables.

Basis for all described iterative methods are the slope-deflection equations or, in general, the equations of the displacement method. Originally they were derived intuitively, by purely mechanical considerations without the proof of convergence and estimation of error, but it can be shown (52,67,77–79) that in principle they are identical to the well-known *mathematical iterative methods* for solution of the system of linear algebraic equations. Thus, the method of moment distribution corresponds to the Gauss-Southwell iteration, the Kani method and the distribution of deformations correspond to the Gauss-Seidel iteration of slope deflection equations. It does not matter here if it is operated with corresponding moments instead of direct rotations.

There is no doubt that purely mathematical representation of iterative methods for frames as an iteration of a system of linear equations is more correct and theoretically precise. It enables us to find the answer to the basic questions of *convergence* (36,67,79,79a) with its conditions and of limits for the values of *errors*, the problems of which were already solved in mathematics. At the same time, it is also possible to give serious study to mathematical properties of the system of equations for frame analysis (80). However, in spite of this, the classical, mechanically interpreted forms of these methods are rather illustrative in mechanical understanding of the procedure and permit an easy orientation.

Space frames and *space behavior* of plane frames, although the number of unknowns is large, can now be solved with help of electronic computers by some of the foregoing methods in matrix form (35,51), if the case of St. Venant's simple torsion in bars can be assumed. Let us note that plane loading of plane frames produces space behavior too, if the bars are subjected to *skew bending*, i.e., the principal axis of inertia of sections of bars are not perpendicular to the plane of the frame. However, in this case a satisfying approximate analysis, by introduction of substitute rigidities for skew bending and of factors of transverse deformation (81), can be done separately for a plane and lateral behavior. An analogous effect, which can be treated similarly, appears at frames having *skew supports* or *skew connections* of bars, the reactions of which do not lie in the plane of the frame. Of great practical importance are the bridge frames with *transversely curved beams* (82).

An analogous problem to that of laterally loaded frames is represented by grid frameworks, which will not be referred to here. We shall only mention that for beams sufficiently closely distributed in grillage, the solution can be transferred to a continuous problem (83,84) replacing the grillage by an anisotropic plate (partial differential equation), which method has not yet been quoted because of smaller importance in the plane behavior of the frame. Solution of frame circular radial grid framework interacting with frame columns was given in (85) where the grid framework was replaced by semi-continuum (ordinary differential equation), formed by continuously distributed radial girders and discrete circular girders.

With respect to the behavior of structural frames further problems are arising. The influence of *semi-*

rigid elastic connection of bars in *joints* (incomplete joint rigidity) can be considered by easy enlargement of ordinary methods. In the displacement method either relative rotation of the end of the bar and of the core of the joint can be considered (34,35,86,130). Otherwise, this case can be interpreted also as a limiting case of bar with various sections, the stiffness of which tends to infinitely small value on an infinitely short portion of bar at the end. The *interaction* of frames and of *elastic subsoil* (86) can also be approximately treated by described methods. The same may be stated of the interaction of arches and upper frames, of the interaction of frames and suspension cables (suspension bridges), of the influence of prestressing cables in frames, and of temperature and shrinkage influences.

Much more complicated is the interaction of frames and of a system of plates and infill walls (88–90), where certain approximate idealizations need to be done, e.g., diagonal strut replacing the infill (91) or panels interconnected in edges.

To make the review complete, the use of *influence lines* should be mentioned here. They were introduced in concept in 1868 by Winkler (32,36,37) and developed mainly by Müller-Breslau. The influence line represents certain static or kinematic quantity of the structure as a function of the position of moving load. It is mostly used to determine extreme effects of moving loads. Their determination is not a problem different from problems previously treated, since it can be transformed to usual stress and strain analysis, if the principle of virtual [or complementary virtual] works (or Maxwell's theorem) is applied. The values of influence line of a certain force [displacement] are then equal to the values of displacement (i.e., deflection line) in the sense of moving unity load which are caused by a displacement [force] of minus unity [unity] value in the sense of the considered force [displacement]. Otherwise, they can be determined also directly as a function, analyzing the frame with general parameter of load position, or by computing their numerical values for various load positions, in which case the use of inverse flexibility or stiffness matrix or of an orthogonalized set of redundants is better than an iterative procedure which would need to be entirely repeated for each load position. Effect of arbitrary loading on arbitrary force or displacements in a frame is easily obtained by combining unit loads and the influence lines of redundants according to the principle of superposition. From the standpoint of mathematical solution of the differential equation of bending (or torsion, tension, shear) of bars, the set of influence lines represents Green's function or its derivatives.

Below, without exhaustive references, are sketched briefly the problems arising if the usual classical theoretical and physical basic assumptions are not sufficient. These problems lie on the border of the classical meaning of the frame analysis and are basically discussed in other subjects, but they have very great importance in modern trends of frame analysis.

The greatest problem which appears with respect to space behavior, especially to torsion, is the admissibility of basic *deformational hypothesis* for the reduction of a three-dimensional problem to one-dimensional. The usual reduction hypothesis used in engineer's solution for bending of bars is the Bernoulli-Navier hypothesis of preserving plane and perpendicular cross sections. In the case of torsion, the case of St. Venant's simple (pure) torsion with zero normal stresses is usually assumed. However, especially in the case of *frames of thin-walled bars,* the latter assumption is not sufficient and the combined (nonuniform) simple and bending (warping) torsion is necessary to be considered (for survey of problem see Nowinski (92)). For bars of open cross section Wagner's assumption of zero shear strain in the middle surface of walls is then made. For open as well as for closed sections, the Umanskij assumption of deplanation (warping), proportional to that for simple torsion, can be done. Besides internal moments, shear and normal forces, rotations and displacements, it is necessary to introduce further quantities, which do not represent static resultants of stress in cross section or displacement of the section as a rigid body, i.e., the bimoments, the rate of twisting rotation. Also it should be emphasized that the difference between centroidal and shear center line of bars needs to be respected. Furthermore, in some cases the completely rigid form of the cross sections cannot be assumed and their distortion should be taken into account (64).

In a mathematically rigorous manner the stresses and strains according to mentioned deformation hypothesis can be regarded as first terms of infinite series for the solution of the three-dimensional elasticity problem by the variational method (64,92). Eventually further terms (self-equilibrating stress systems) also can be introduced in order to fulfil the compatibility conditions. Here, of course, the question of optimum selection of these deformation hypotheses is of importance.

The solution of *space behavior* of frames whose bars are subjected to warping torsion (92–101) or eventually also to distorsion of sections, is relatively much more complicated and further work remains to be done. The analogic method to the transfer matrices method, or to the force and displacement methods, can be used for analysis. Especially careful attention should be paid to the behavior of joints, to their stiffening against warping, to the transfer of bimoments and of deplanation (warping) of sections through joints and so on. It should be mentioned that there are joints transferring the deplanation entirely, partially or not at all.

To obtain correct results, it is often necessary to introduce *second order theory*, which considers the equilibrium on the deformed form of frame and represents a strength approach to the stability problem. This is important not only for flat arches and suspension bridges, which will not be discussed here, but also for building frames with slender columns, where the moments of normal force to the deflected centroidal axis can have considerable influence (102). The stiffness of bar then depends on acting forces, so that the problem becomes nonlinear. For instance, the displacement method (102, 105) and Kani's method (103), moment distribution (104) and relaxation method (36, Chap. 9.5) can be enhanced for this purpose.

If the deflections of bars are considerably great, then it is necessary to consider *finite deformations*, especially the changes of length of bars caused by bending (106). This also influences the second order analysis and stability of frames (107).

In the second order theory or in the theory of finite deformations the problem becomes nonlinear, either statically or geometrically, respectively, and the principle of superposition is no more valid. The same follows if the material of frame is physically nonlinear, i.e., *nonlinearly elastic*. Using the force method or the theorem of complementary energy, a system of nonlinear algebraic equations is obtained. If the displacement method or the theorem of the strain energy is used, the problem leads to a system of first-order nonlinear ordinary differential equations with load parameter as independent variable (108). Also various successive approximations by series of linear analysis can be used (109). For reinforced concrete frames (110), the nonlinearity of concrete is complicated by the forming of cracks and yielding of reinforcement.

Limiting case of material nonlinearity is the *elasto-plastic* and the *rigid-plastic stress-strain* diagram, which can be well accepted for steel. Limited analysis of frames with determination of collapse load has been largely worked out (111–118). More accurately in limit analysis the strain hardening should be respected (117). For limit analysis of concrete frames it is necessary to consider the limited rotation capacity of plastic hinges (119,120, 121). Recently great attention has been paid to the *optimum design* (minimum weight design) of frames which is based on plastic analysis (122–124).

In the presence of *creep*, time should be taken into account. For building frames the creep of concrete (regardless of creep of prestressing cables), which is not simply visco-elastic with respect to aging, is of importance. Creep affects the increase of deformations and the redistribution of stresses in frame which can be caused by initial state, e.g., the process through which the frame was cast or the prestressing, by shrinkage or temperature dilatations, by nonhomogeneity, i.e., concrete-steel, differences in age, dimensions and moisture of concrete, and by nonlinearity of creep itself for concrete at high stresses. In analysis of creep of concrete frames the use of the force method is generally easier than that of the displacement method. For static redundants it follows a system of ordinary differential equations or a system of Volterra's integral equations which are linear, if the creep is linear (125,126,128). The solution can be transformed to a series of linear elastic analyses done by any of the foregoing methods (125,126). The presence of creep influences further the second order analysis, the long-time stability, finite-deformations theory and others. The time parameter also appears for frames founded on creeping soil, i.e., on clay layer. Its creep can be eventually combined with the creep of frame (127).

The individual types of nonlinear effects, time effects, deformation hypotheses and so on, can in reality occur also simultaneously. However, the analysis respecting combination of these effects is rather complicated and further work remains to be done.

All problems referred to up to now concern static stress and strain analysis of frames. The closely related problems of stability (36,104,105,129,130), dynamics and vibrations (104,131) of frames can be analyzed by further enlargement of the discussion, but their review would need special separate surveys.

The following list of references does not claim to be exhaustive and we are aware that some important contributions from the extremely vast literature on this subject may have been omitted.

REFERENCES

1. A. Castigliano, *Théorie de L'équilibre des Systémes Élastiques*, Turin, 1879.

2. E. Robert and L. Musette, *Le Calcul des Systèmes Hyper-Statiques*, Liège, 1945.

3. F. Stüssi, Zur Berechnung des Vierendeelträgers, AMR, 4, Rev. 3542.

4. C. Culmann (prepared for publication by W. Ritter), *Anwendungen der Graphischen Statik, III*, Zürich, 1900.

5. E. Suter, *Die Methode der Festpunkte*, 3rd ed., Berlin, 1951.

6. J. Cl. Maxwell, "On the calculation of the equilibrium and stiffness of frames," *Phil. Mag.*, 1864.

7. H. Manderla, "Jahresbericht der Techn.," *Hochschule in München*, 1878-1879.

8. E. Winkler, *Theorie der Brücken, II*, Wien, 1881.

9. O. Mohr, "Die Berechnung der Fachwerke mit starren Knotenverbindungen," *Der Civilingenieur*, 1892-1893.

10. A. Ostenfeld, *Die Deformationsmethode*, Berlin, 1926.

11. R. Guldan, *Rahmentragwerke und Durchlaufträger*, 5th ed., Wien, 1954.

12. Hardy Cross, "Analysis of continuous frames by distributing fixed-end moments," *Proc. Amer. Soc. Civ. Engrs.*, May 1930.

13. Hardy Cross and N. D. Morgan, *Continuous Frames in Reinforced Concrete*, New York, 1932.

14. V. Dašek, "Das allegemeine Kräfte und Momentenverteilungsverfahren," *Beton u. Eisen*, 1941.

15. G. A. Maney and W. M. Wilson, "Wind stresses in steel frames of office buildings," *Univ. Ill. Engng. Exp. Sta. Bull.*, no. 80, 1915.

16. L. E. Grinter, "Analysis of continuous frames by balancing angle changes," *Proc. Amer. Soc. Civ. Engrs.*, 1936.

17. C. Klouček, *Das Prinzip der fortgeleiteten Verformung*, Berlin, 1941.

18. C. Klouček, "Structural analysis by distribution of deformation," AMR, 4, Rev. 4143.

19. R. V. Southwell, *Relaxation Methods in Engineering Science*, Oxford, 1940.

20. Fr. Bleich, *Die Berechnung statisch unbestimmter Tragwerke nach der Methode des Viermomentensatzes*, Berlin, 19:8.

21. L. C. Maugh, *Statically Indeterminate Structures*, New York, 1946.

22. L. A. Beaufoy, "Vierendeel truss analysis using equivalent elastic systems," AMR, 6, Rev. 449.

23. L. A. Beaufoy, "Open-spandrel arch analysis assuming continuity of structure," AMR, 7, Rev. 2831.

24. R. W. Stewart, "Analysis of frames with elastic joints," *Trans. Amer. Soc. Civ. Engrs.*, 1939.

25. R. W. Stewart and A. Kleinlogel, "Die Traversen-Methode," AMR, 6, Rev. 832.

26. J. C. Richards, "Stress-determination for a three-dimensional rigid jointed framework by the method of systemic relaxation of constraints," *J. Inst. Civ. Engrs.*, 1937.

27. Z. Bažant, "Les portiques dans l'espace," AMR, 5, Rev. 2814.

28. J. H. Argyris, "On the analysis of complex elastic structures," AMR, 11, 7, 1958, pp. 331-338.

29. J. H. Argyris, "Energy theorems and structural analysis," *Aircr. Engng.*, 26, no. 10, 1954, p. 347; no. 11, p. 383; v. 27, no. 2, 1955, p. 42; no. 3, p. 80; no. 4, p. 125; no. 5, p. 145; AMR, 10, 1957, Rev. 54.

30. J. H. Argyris, "Die Matrizentheorie der Statik," *Ingen. Arch.*, 25, 3, p. 174; AMR, 11, 1958, Rev. 81.

31. S. O. Asplund, "Indeterminate analysis," *Publ. Internat. Assoc. Bridge Struct. Engng.*, 21, 1, 1961.

32. K. Beyer, *Die Statik im Stahlbetonbau*, 2nd ed., Berlin, Springer, 1956.

33. V. Dašek, *Statics of Framed Structures* (in Czech) Prague, Czech. Acad. Sci. Publ., 1959.

34. A. S. Hall and R. W. Woodhead, *Frame Analysis*, John Wiley and Sons, New York, 1961.

35. S. Shore, "The elements of matrix structural analysis," *Proc. 2nd Conf. on Electronic Comp. Amer. Soc. Civil Engrs.*, 1960, p. 145.

36. J. A. L. Matheson, N. W. Murray and R. K. Livesley, *Hyperstatic Structures*, London, Butterworth, 1959.

37. J. I. Parcel and R. B. B. Moorman, *Analysis of Statically Indeterminate Structures*, John Wiley and Sons, New York, 1955.

38. R. G. Robertson and co-workers, "Portal frame analysis by moment area method," *Struct. Engr.*, 34, 5, 1956, pp. 173-178.

39. J. J. Tuma and J. T. Oden, "String polygon analysis of frames with straight members," *Proc. Amer. Soc. Civil Engrs.*, 81, ST7 (*J. Struct. Div.*), October 1961, pp. 63-89; AMR, 15, 1962, Rev. 1461.

40. K. Havelka, J. Harvančik and J. Trokan, "Framed bridge and industrial building structures solved by the method of fictional statical conditions (in Slovak)," Bratislava, Slov. Acad. Sci. Publ., 1959; AMR, 16, 1963, Rev. 2216.

41. J. Kiusalaas, S. L. Lee and M. Makino, "Analysis of continuous space frames by conjugate frame analogy," *Struct. Engr.*, no. 6, 1963, pp. 203-208.

42. Z. P. Bažant, "Spatial force-deformation analogy for beams and frames," to be published in *Proc. Amer. Soc. Civil Engrs.*

43. W. L. Miranker, "Applications of topology to analysis," AMR, 16, 1, 1963, p. 1.

44. G. Kron, "Tearing and interconnecting as a form of transformation," *Quart. Appl. Math.*, 13, 2, July, 1955, pp. 147-159; AMR, 9, 1955, Rev. 356.

45. E. Dony and J. Goethals, "Méthode matricielle de calcul des structures hyperstatiques," *Acier-Stahl-Steel*, Bruxelles, no. 1, 1959.

46. N. C. Lind, "Analysis of structures by system theory," *Proc. Amer. Soc. Civil Engrs.*, 88, ST2 (*J. Struct. Div.*), April, 1962, pp. 1-22; AMR, 16, 1963, Rev. 3391.

47. W. R. Spillers, "Application of topology in structural analysis," *Proc. Amer. Soc. Civil Engrs.*, 89, ST4 (*J. Struct. Div.*), August 1963, pp. 301-343.

48. F. Di Maggio, "Statical indeterminacy and stability of structures," *Proc. Amer. Soc. Civil Engrs.*, 89, ST3 (*J. Struct. Div.*), June 1963, pp. 63-76.

49. S. J. Fenves and F. B. Branin, "Network-topological formulation of structural analysis," *Proc. Amer. Soc. Civil Engrs.*, 89, ST4 (*J. Struct. Div.*), August 1963, pp. 483-514 (Discussions February 1964, p. 261).

50. F. Schleicher (ed.), *Taschenbuch für Bauingenieure*, v. 1, Chap. 4 by E. Kohl, Berlin, Springer, 1955.

51. G. Grünning and A. Hütter (eds.), *Ingenieurtaschenbuch Bauwesen*, v. 1, Chap. 4 by G. Grünning, Chap. 5 by M. Müller, Leipzig, Teubner, 1963.

52. A. J. Bignoli, "Analysis de estructuras aporticadas," Spanish translation with comments of survey by Z. Bažent, "Analysis of framed structures," AMR, 8, March, 1955; *Cienc. Téc.*, 125, 1958, pp. 263, 630.

53. L. M. Pei, "Stiffness method of rigid frame analysis," *Proc. 2nd Conf. on Electronic Comp. Amer. Soc. Civil Engrs.*, 1960, p. 225.

54. E. Lightfoot and A. Sawko, by electronic computer," *Engineering*, 187, January 1959, p. 18; AMR, 12, 1959, Rev. 6116.

55. V. K. Krylov, "On the analysis of frames with great number of nodes" (in Russian), *Symposium Strojit. mech. i. raschet konstruktsii*, no. 2, Moscow, Gostrojizdat, 1961, p. 22.

56. Z. P. Bažant, "Analysis of influence lines of continuous frames with hinges at midspans" (in Czech), *Inženýrské stavby*, Prague, 9, 9, 1961, p. 344.

57. J. Courbon, "Calcul des ponts à poutres consoles réunies par des articulations," *Publ. Internat. Assoc. Bridge Struct. Engng.*, 17, 1957, p. 9.

58. S. O. Asplund, "Inversion of band matrices," *Proc. 2nd Conf. of Electronic Comp. Amer. Soc. Civil Engrs.*, 1960, p. 513.

59. Ju. G. Plechkin, "Inverse matrix of the systems of three-

terms equations" (in Russian), *Symposium Issledovanija po teorii soorushenij*, Moscow, Gostroiizdat, 1961, p. 253.

60. V. D. Shaikevich, "Matrix method of analysis of regular frame systems" (in Russian), *Symposium Raschet prostranstvennych konstruktsii*, no. 4, Moscow, Gosstroiizdat, 1958.

60a. H. S. Tsang, "Analysis of rigid frames by difference equations," *Proc. Amer. Soc. Civil Engrs.*, 89, ST 2 (*J. Struct. Div.*), April 1963, pp. 127-159.

60b. E. Keintzel, "Die Berechnung axialsymmetrischer räumlicher Stockwerkrahmen mit Hilfe von Differenzengleichungen," *Rev. Mecan. Appl.* (Bucuresti), no. 4, 1964, p. 923.

61. E. Lightfoot, "Substitute frames in the analysis of rigid jointed structures," *Civil Engng.*, London, 52, December 1957, p. 1381; 53, January 1958, p. 70; AMR, 11, 1958, Rev. 3978.

62. A. Aass, "Matrix progression method," *Bauingenieur*, 39, no. 8, p. 306; no. 10, 1964, p. 417.

63. S. Falk, "Die Berechnung offener (geschlossener) Rahmentragwerke nach dem Reduktionsverfahren," *Ingen.-Arch.*, no. 1, 1958, p. 61; no. 2, p. 109; AMR, 11, 1958, Rev. 4951.

64. V. Z. Vlasov, *Thin-Walled Elastic Beams*, 2nd ed., Moscow, Fizmatgiz, 1959; AMR, 13, 1960, Rev. 2739; Engl. transl., AMR, 16, 1963, Rev. 5719.

65. G. Kani, *Die Berechnung mehrstöckiger Rahmen*, Stuttgart, Wittwer, 1956; Engl. transl., AMR, 12, 1959, Rev. 1298.

66. M. F. Rubinstein, "Multistory frame analysis by digital computers," *Proc. 2nd Conf. Electronic Comp. Amer. Soc. Civil Engrs.*, September, 1960, p. 261.

67. P. Lustgarten, "Iterative method in frame analysis," *Proc. Amer. Soc. Civil Engrs.*, 89, ST2 (*J. Struct. Div.*), April 1963, pp. 75-94.

68. F. P. Weisinger, S. L. Lee and D. L. Guell, "Analysis of arch frames," *Proc. Amer. Civil Engrs.*, 89, ST5 (*J. Struct. Div.*), October 1963, pp. 15-33.

69. J. Fader, "Grid analysis by the reaction distribution method," *Proc. Amer. Civil Engrs.*, 87, ST6 (*J. Struct. Div.*), August 1961, pp. 77-103; AMR, 15, 1962, Rev. 2697.

70. D. V. Reddy and A. W. Hendry, "Rapid moment and torque distribution method for grid framework analysis," *Civil Engng.*, 54, London, July-August, 1959, p. 867.

71. P. Csonka, "Une contribution à la simplification de la méthode de Hardy Cross," *La Technique Moderne-Construction*, March 1952, p. 85; AMR, 6, 1953, Rev. 86.

72. S. Zaytseff, *La méthode de Hardy Cross et ses simplifications*, Paris, Dunod, 1952; AMR, 5, 1952, Rev. 2593.

73. A. Habel, "Zur Windberechnung vielgeschossiger Stockwerkrahmen nach Takabeya," *Bautechnik*, 37, no. 8, 1960, p. 307.

74. P. Csonka, "Beitrag zur Berechnung waagerecht belasteter Stockwerkrahmen," *Bautechnik*, 39, 7, 1962, p. 237.

75. M. Todorov, "Iterationsverfahren mit beschleunigter Konvergenz zur Berechnung von Stockwerkrahmen," *Bautechnik*, 37, 1, 1960, p. 18.

76. Z. P. Bažant, "Relaxation with variable load term, and its application in solving plates and torsion problems" (in Czech), *Apl. Mat. Ceskoslov. Akad. Ved.*, 5, 6, 1960, pp. 458-472; reprinted in English in *Abstracts of Selected Articles from Soviet Bloc and Mainland China Technical Journals*, ser. 1, no. 16, May 1962, U. S. Dept. of Commerce, OTS, Washington 25, D. C.; AMR, 16, 1963, Rev. 711.

77. H. Müller, "Die Verfahren von Cross und Kani als Anwendungen der üblichen Methoden der iterativen Auflösung von Gleichungssystemem," *Bauplanung-Bautechnik*, 12, 2, 1958, p. 118.

78. J. Mai, Über die Lösung linearer Gleichungssysteme in der Baustatik," *Bautechnik*, 40, 3, 1963, p. 83.

79. K. Chobot, "Matrix form of distribution method" (in Czech), *Práce ČVUT*, ser. I, no. 2, Prague, State Pedagogical Publ. House, 1963, p. 33.

79a. Ja. B. Lvin, "Analysis of complicated frames by the matrix iteration method" (in Russian), *Symposium Issledovaniya po teorii sooruzheniy*, 13, Moscow, Gosstroyizdat, 1964, pp. 217-229.

80. I. Babuška and M. Fiedler, "Über Systeme linearer Gleichungen vom Typ der Rahmentragwerke," *Apl. Mat. Československ. Akad. Ved.*, 4, 6, 1959, p. 441.

81. Z. P. Bažant, "Analysis of framed structures with bars subjected to skew bending" (in Czech), *Inženýrské stavby*, Prague, 9, 6, 1961, pp. 225-228.

82. Z. P. Bažant, "Influence lines of horizontally curved bridges" (in Czech, Engl. summ.), *Stavebnícky Časopis*, 12, 1, pp. 18-39.

83. Ch. Massonet, "Méthode de Calcul des ponts à poutres multiples tenant compte de leur résistence à la torsion," *Publ. Internat. Bridge Struct. Engng.*, 19, 1959, p. 147.

84. R. Bareš, *Analysis of Grid Frameworks with Respecting of Torsion* (in Czech), Prague, State publ. house techn. lit., 1963.

85. J. Jiroušek, "Circular beam grillages and space rotation frames," Proc. Techn. Univ. of Brno, no. 1-4, 31, Prague, State pedagog. publ. house, 1963.

86. G. R. Monforton and T. S. Wu, "Matrix analysis of semi-rigidly connected frames," *Proc. Amer. Soc. Civil Engrs.*, 89, ST6 (*J. Struct. Div.*), 13, December 1963, pp. 13-42.

87. D. Augustin, "Rahmenknickung bei elastischer Einspannung im Baugrund," *Bauingenieur*, 36, 12, 1961, p. 441.

88. B. Cardan, "Concrete shear walls combined with rigid frames in multistory buildings subject to lateral loads," *J. Amer. Concrete Inst.*, 58, 3, pp. 299-316; AMR, 15, 1962, Rev. 3365.

89. R. Rosman, "Beitrag zur Untersuchung des Zusammenwirkens von waagerecht belasteten Wänden und Stockwerkrahmen bei Hochbauten," *Beton u. Stahlbetonbau*, 58, 2, 1963, pp. 36-40.

90. W. W. Frischmann and S. S. Prabhu, "Multistory frames and interconnected shear walls subjected to lateral loads, *Concrete Construct. Engng.*, 58, 6, 1963, pp. 227-234.

91. B. S. Smith, "Lateral stiffness of infilled frames," *Proc. Amer. Soc. Civil Engrs.*, 88, ST6 (*J. Struct. Div.*), December 1962, pp. 183-199; AMR, 16, 1963, Rev. 4596.

92. J. Nowinski, "Theory of thin-walled bars," AMR, 12, 3, 219, 1959.

93. B. N. Gorbunov, "Analysis of three-dimensional frames composed of thin-walled bars" (in Russian), *Prikl. Mat. Mekh.*, 7, 1, 1943, p. 65.

94. B. N. Gorbunov and A. I. Stryelbitskaya, *Theory of Frameworks Composed of Thin-Walled Bars* (in Russian), Moscow, Gostekhteorizdat, 1948.

95. D. V. Bytchkov, *Analysis of Beam and Frame Systems Composed of Thin-Walled Bars* (in Russian), Moscow, Gosstroyizdat; 1948.

96. N. J. Karyakin, "Torsion of thin-walls bars and frames" (in Russian), *Memiit*, Moscow, 1950.

97. L. N. Stavraki, *Stability of Space Frames of Thin-Walled Bars with Open Symmetrical Cross Section* (in Russian), Sbornik Inst. Stroit. mekh., Kiev, Akad. Sci. U.S.S.R., no. 12, 1950.

98. L. N. Stavraki, "Strength analysis of three-dimensional structures composed of thin-walled bars with open symmetrical profile" (in Russian), *Raschet Prostanstvennykh Konstruktsii*, no. 2, Moscow, Gosstroiizdat, 1951, p. 172.

99. J. V. Urban, *Theory of Analysis of Thin-Walled Bar Structures* (in Russian), Moscow, Gosstroyizdat, 1955.

100. R. K. Livesley and D. B. Chandler, *Stability Functions for Structural Frameworks*, Univ. Press, Manchester, 1956.

101. J. D. Renton, "Stability of space frames by computer analysis," *Proc. Amer. Soc., Civil Engrs.*, 88 ST4 (*J. Struct. Div.*), August 1962, pp. 81-103.

102. H. Kretzschmar and H. Müller, "Berechnung von Stockwerkrahmen nach Theorie II. Ordung" Prelim. Publ., *7th Congress Internat Assoc. Bridge Struct. Engng.* in Rio de Janeiro, 1964, p. 75.

103. K. Kloeppel and H. Ebel, "Beitrag zur Berechnung von

Stockwerkrahmen auf Stabilität und nach der Spannungstheorie II. "Ordnung," Stahlbau, 31, 1, January 1962, pp. 1-7; AMR, 15, 1962, Rev. 6361.

104. S. Blaszkowiak and Z. Kaczkowski, *Metoda Crossa* (in Polish), 3rd ed., Warszawa, Polskie Wydaw. Naukowe, 1964.

105. Z. Bažant, "Buckling strength of framed structures" (in Czech), Technicky Obzor, Prague, 51, 1943, pp. 7-8, 16, 97-102, 124-127, 255.

106. W. Merchant and D. M. Brotton, "A generalized method of analysis of elastic plane frames," Prelim. publ. *7th Congress Internat. Assoc. Bridge Struct. Engng.* in Rio de Janeiro, 1964, p. 87.

107. M. R. Horne, "Effect of finite deformations in elastic stability of plane frames," *Proc. Roy. Soc. London*, ser. A, 226, 1324, 1962, pp. 47-67.

108. J. E. Goldberg and R. M. Richard, "Analysis of nonlinear structures," *Proc. Amer. Soc. Struct. Engrs.*, 89, ST4 (*J. Struct. Div.*), August 1963, pp. 333-354.

109. E. J. Wilson, "Matrix analysis of nonlinear structures," *Proc. 2nd Conf. on Electronic Comp. Amer. Soc. Civil Engrs.*, 1960.

110. J. F. Borges, E. Arantes and E. R. Oliveira, "Nonlinear analysis of reinforced concrete structures," *Publ. Intern. Assoc. Bridges Struct. Engng.*, 23, 51, 1963.

111. P. G. Hodge, *Plastic Analysis of Structures*, McGraw-Hill, New York, 1959; AMR, 12, 1959, Rev. 3254.

112. G. Neal, *The plastic methods of structural analysis*, 2nd ed., Chapman & Hall, London, 1963.

113. J. F. Baker, M. R. Horne and co-workers, *The Steel Skeleton*, v. 2, Plastic behaviour and design, University Press, Cambridge, 1956; AMR, 9, 1956, Rev. 3915.

114. Ch. Massonnet and M. Save, "Calcul plastique der constructions," Centre Belgo-Lux. d'Information de l'Acier, Bruxelles, 1961; AMR, 15, 1962, Rev. 2114.

115. J. Heyman, "On the estimation of deflections in elastic-plastic framed structures," *Proc. Instn. Civil Engrs.*, 19, no. 5, 1961, pp. 39-60; AMR, 15, 1962, Rev. 1970.

116. M. D. Hangan, "Le calcul des structures hyperstatiques dans le domaine plastique," *Ann. Inst. Techn. Bat. Trav. Publ.*, 16, no. 183/184, 1963, pp. 299-324.

117. J. Heyman, "Limit design of framed structures," *Engineering*, 187, no. 4851, February 1959, p. 275.

118. "Symposium on plastic theory of structures," *Brit. Welding J.*, 3, no. 8, August 1956, pp. 331-378.

119. A. L. L. Baker, "A general analysis of elasto-plastic three-dimensional frames," *Publ. Internat. Assoc. Bridge Struct. Engng.*, 20, 1960, pp. 1-18; AMR, 15, 1962, Rev. 5881.

120. A. L. L. Baker, "Ultimate load theory for concrete frame analysis," *Transactions Amer. Soc. Civil Engrs.*, 27, part II, 1962.

121. "Ultimate load design of concrete structures," Report of Research Committee, Proc. Instn Civil Engrs., 21, February 1962, pp. 399-442.

122. J. Heyman, "On absolute minimum weight design of framed structures," *Quart. J. Mech. Appl. Math.*, 12, 3, August 1959, pp. 314-324; AMR, 13, 1960, Rev. 3997.

123. G. J. Megarefs and P. G. Hodge, "Method for plastic design of frames," *Proc. Amer. Soc. Civil Engrs.*, 89, ST 1 (*J. Struct. Div.*), February 1963, pp. 197-211.

124. R. K. Livesley, "Optimum design of structural frames for alternative systems of loading," *Civil Engng.* (London), 54, no. 636, June 1959, pp. 737, 739-740.

125. Z. P. Bažant, "Die Berechnung des Kriechens und Schwindens nicht homogener Betonkonstruktionen," *Publ. Prelim. 7th Congress Internat. Assoc. Bridge Struct. Engng. in Rio de Janeiro*, 1964, p. 887.

126. Z. P. Bažant, *Creep of Concrete in Structural Analysis* (in Czech), Prague, State publ. house techn. lit. (in press 1965).

127. Z. P. Bažant, "Time interaction of structures and subsoil" (in Czech), *Stavebnícky casopis*, 12, 9, 1964, pp. 542-558.

128. N. Kh. Arutyunyan, *Some Questions of Creep Theory* (in Russian), Moscow, Gostechteorizdat, 1952.

129. F. Bleich, "Buckling strength of metal structures," McGraw-Hill, New York, 1952; AMR, 5, 1952, Rev. 1037.

130. V. Fiŕt, *Stability of Precast Structures* (in Czech), Prague, State Publ. House Techn. Lit., 1963.

131. V. Koloušek, *Calcul des Efforts Dynamiques dans les Ossatures Rigides*, Paris, Dunod, 1959.