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Friction and Cracking in Constitutive Modeling of Geomaterials

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SYNOPSIS Attention is focused on mathematical modeling of the inelastic behavior of geomaterials caused by progressive development of microcracks as well as friction on crack surfaces in contact. First the problem of incremental nonlinear triaxial constitutive equations reflecting internal friction is discussed from the viewpoint of work inequalities which generalize the well-known Drucker's postulate. Subsequently, the progressive development of microcracks which leads to strain-softening and unstable strain localization is analyzed. Presented is a new model in which the formation and propagation of fracture through progressive microcracking is modeled by means of stress-strain relations for a crack band of a single element width. Numerical applications in finite element analysis are outlined and comparisons with existing test data on Mode I fracture of rock are indicated. It is shown that this approach can correctly capture the deviations from classical linear fracture mechanics predictions, including the R-curves. Finally, certain important limitations of orthotropic incremental stress-strain relations for materials undergoing progressive damage are pointed out and their implications are analyzed.

INTRODUCTION

In contrast to metals or polymers, one major difficulty in the mathematical modeling of the inelastic behavior of geomaterials is the description of the deformations caused by progressive development of microcracks and by friction on crack surfaces in contact. Except under conditions of high hydrostatic pressure, the inelastic deformation of these materials is attributable principally to the opening and slip of microcracks and not to plastic slip whose source are dislocations in a crystallic lattice. The reason for this type of behavior is the internal heterogenity of these materials coupled with the brittleness of the binding matrix and bond between the grains in the materials. Such characteristics are typical of various rocks, as well as the artificial rock — concrete. Moreover, the frictional aspects are characteristic also of granual soils.

In the present lecture various new approaches to the mathematical modeling of these materials recently developed at Northwestern University will be outlined and their relationship to other modeling approaches will be analyzed. Attention will be centered on models applicable to finite element analysis. Comparisons with existing experimental data on rock fracture will be shown and some numerical finite element results will be demonstrated.

FRICTION AND DILATANCY

Friction causes a major complication in constitutive modeling since it leads to violation of the basic stability postulate, namely Drucker's postulate [9-20], which serves as the basis for the flow rule (normality rule). This postulate is given by the inequality

$$\Delta W = \frac{1}{2}\, d\sigma_{ij}\, d\varepsilon_{ij}^{p\ell} > 0 \qquad (1)$$

in which ΔW represents the second order work dissipated during a cycle of applying and removing stress increments $d\sigma_{ij}$ and $d\varepsilon_{ij}^{p\ell}$ are the increments of plastic strains. This postulate is known to represent a sufficient but not necessary condition for local stability of the material [9, 10, 13, 19, 21]. It is important to realise that its violation does not necessarily imply instability. In fact, certain conditions under which this postulate may be violated yet stability is still guaranteed has been recently formulated [4], extending previous analysis by Mandel [21]. The following more general inequality which guarantees stability of the material has been found [4]:

$$\Delta W - \chi\, \Delta W_f > 0 \qquad (2)$$

in which χ is a parameter which can have any value between 0 and 1. Drucker's postulate is obtained for $\chi = 0$. The quantity ΔW_f represents a second-order energy expression defined as

$$\Delta W_f = \frac{\beta' - \beta^*}{2C}\, d\sigma\,(d\bar{\tau} + \beta^* d\sigma) \qquad (3)$$

which may be called the frictionally blocked elastic energy. The material parameters in Eq. 3 are defined as

$$C = -k_1 \frac{\partial F/\partial \gamma^{p\ell}}{\partial F/\partial \bar{\tau}},\quad \beta' = \frac{\partial F/\partial \sigma}{\partial F/\partial \bar{\tau}},\quad \beta^* = k_2 \frac{d\varepsilon^{pv}}{d\gamma^{p\ell}} \qquad (4)$$

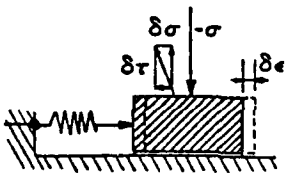


Fig. 1 - Example of a Spring Loaded Frictional Block

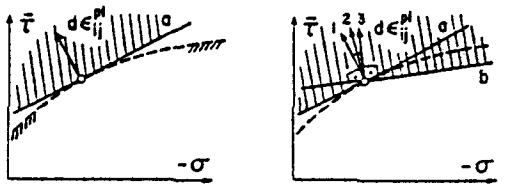


Fig. 2 - Plastic Strain Increment Relative to Loading Surface [4].

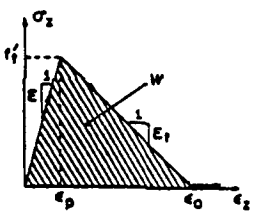
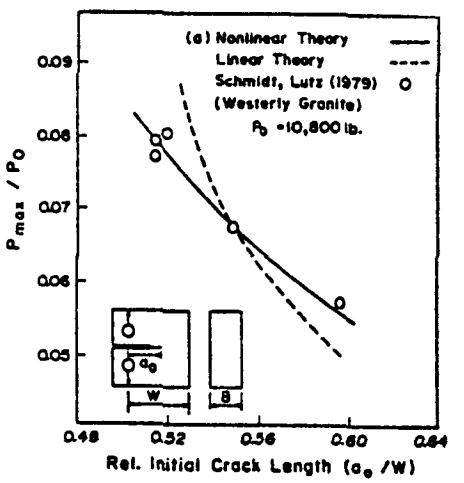


Fig. 3 - Uniaxial Stress-Strain Relation Used in the Present Non-linear Fracture Model



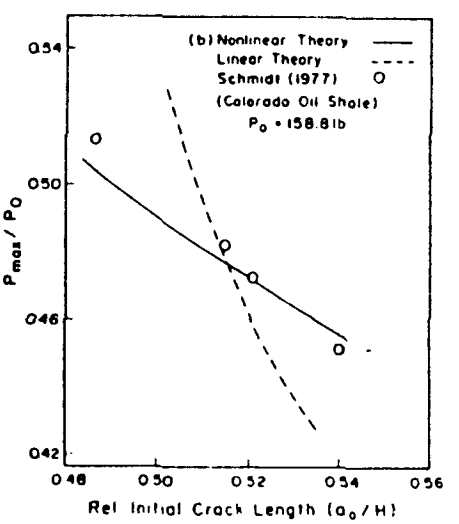


Fig. 4 - (a) Fits of the Maximum Load Data of Schmidt and Lutz (1979) for Westerly Granite, and (b) the Same for Data of Schmidt (1977) For Colorado Oil Shale.

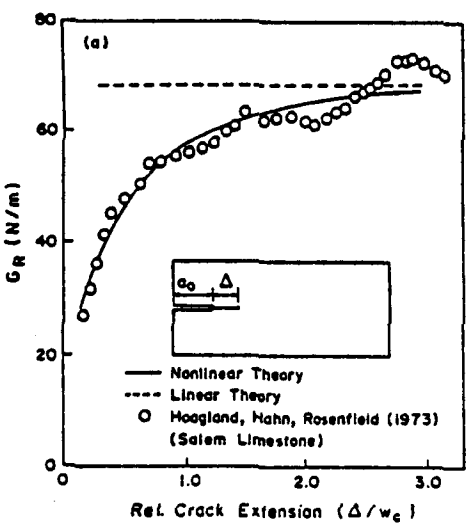


Fig. 5 - Apparent Fracture Energy vs. Crack Extention According to Hoagland et al. (1973) for Salem Limestone, and the Fit by Present Nonlinear Theory.

where

$$F(\sigma, \bar{\tau}, \hat{\gamma}^{p\ell}) = 0 \qquad (5)$$

represents the loading function of the material. The variable σ represents the mean (hydrostatic) stress, $\bar{\tau}$ is the stress intensity (square root of the deviator of the strain tensor ε_{ij}), and $\hat{\gamma}^{p\ell}$ is the length of the path traced in the space of plastic strains, which is used as a hardening parameter.

The meaning of the material parameters in Eq. 3 may be illustrated taking recourse to Mandel's example [21](Fig. 1). A frictional block, resting on a rough horizontal surface, is considered loaded by a vertical force simulating the hydrostatic stress σ, and is also subjected to force F from a horizontal spring such that the sliding of the block is imminent. A horizontal force applied on the block simulates $d\bar{\tau}$. Mandel showed that if a disturbing force $d\sigma_{ij}$ inclined from the vertical to the left by an angle less than the friction angle is applied (Fig. 1), the block is caused to slide but since the displacement to the right is infinitesimal, the initial equilibrium state of the block is stable. Yet, Drucker's postulate (Eq. 1) is violated for this displacement. Inequality (2) is not violated. Coefficient from this inequality represents the spring constant, coefficient β' is the friction coefficient of the block, and β^* is the dilatancy angle indicating the ratio of lifting of the block to its sliding.

The new generalized condition sufficient for material stability (Eqs. 2, 3) makes it possible to formulate a frictional constitutive relation and check whether such a relation satisfies stability conditions. Moreover, by following the same line of reasoning as in classical incremental plasticity, one can derive the flow rule associated with this inequality. Such an analysis shows [4] that this flow rule allows certain, but not arbitrary violations of the normality rule. It is found, for example, that in the plane of $\bar{\tau}$ vs. σ the admissible load increment vectors can deviate from the normal to the right and fill a fan of directions, the limiting inclined direction being uniquely determined by the loading surface (Fig. 2). The resulting flow rule is, however, completely different from that for non-associated plasticity because a single loading surface is used and because, in cotradistinction with non-associated plasticity, stability of the material is guaranteed.

Inequality (2) pertains only to the friction in deviatoric deformations caused by hydrostatic compression. It is possible to derive a more general inequality [4] which also involves the friction in volume change caused by deviator shear stress, a phenomenon which may be called the inverse friction. Further generalizations are possible when, in addition to loading surfaces (potentials) in the stress space one also uses loading surfaces in the strain space.

PROGRESSIVE MICROCRACKING AND FRACTURE

Consider now progressive formation of microcracks caused by stresses σ_x, σ_y, σ_z, the principal directions of which remains fixed. Let the normal stress and strain components be grouped into the column matrices $\underset{\sim}{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$, $\underset{\sim}{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z)^T$, where T denotes the transpose, and ε_x, ε_y, ε_z are the normal strains, assumed to be linearized, or small. The elastic stress-strain relation for the normal stress and strain components may be considered as $\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$ or $\underset{\sim}{\varepsilon} = \underset{\sim}{C}\,\underset{\sim}{\sigma}$

in which

$$\underset{\sim}{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{sym.} & & D_{33} \end{bmatrix} \qquad (6)$$

and

$$\underset{\sim}{C} = \underset{\sim}{D}^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym.} & & C_{33} \end{bmatrix} \qquad (7)$$

Here $\underset{\sim}{D}$ and $\underset{\sim}{C}$ represent the stiffness and compliance matrices of the uncracked material.

To figure the changes in the foregoing matrices caused by progressive microcracking, it is helpful to recall first the stiffness matrix of a fully cracked material, i.e., an elastic material which is intersected by continuously distributed (smeared) cracks normal to axis z; see e.g., Suidan and Schnobrich [26]. This stiffness matrix is known to have the form

$$\underset{\sim}{D}^{fr} = \begin{bmatrix} D_{11} - D_{13}^2 D_{33}^{-1}, & D_{12} - D_{13} D_{23} D_{11}^{-1}, & 0 \\ & D_{22} - D_{23} D_{33}^{-1}, & 0 \\ \text{sym.} & & 0 \end{bmatrix} \qquad (8)$$

This matrix is derived from the condition that the stress normal to the cracks must be 0 and that the material between the cracks has the properties of an uncracked elastic material. This last assumption is of course a simplification, since often the material between continuous cracks may be damaged by presence of discontinuous microcracks which formed before the continuous cracks were produced.

Comparing the matrices in Eqs. 6 and 8, we see that each element of the stiffness matrix is affected by cracking. Thus, modeling of a continuous transition from Eq. 6 to Eq. 8 would not be easy since each term of the stiffness matrix would have to be considered a function of some parameter of cracking. It appears however [6] that the situation becomes much simpler if one uses the compliance matrix $\underset{\sim}{C}$. It appears that one needs to consider only one element of the compliance matrix to change

$$\underset{\sim}{C}(\mu) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym.} & & C_{33}\mu^{-1} \end{bmatrix} \qquad (9)$$

in which μ is a certain cracking parameter. There are two reasons for the effect of cracking on the compliance matrix to have this simple form:

1. If we assume all microcracks to be perfectly straight and normal to axis z, which is of course a simplification (in reality the microcrack orientations exhibit a certain statistical distribution), then formation of the microcrack has no effect on stresses σ_x and σ_y in the directions parallel to the crack planes. Therefore, appearance of microcracks normal to z should have only the effect of increasing the strain in the z direction, which is why, under this assumption, only the third diagonal term in Eq. 9 should be affected.

2. The second reason is that, in the limit $\mu \to 0$, the inverse of the compliance matrix in Eq. 9 is identical to the stiffness matrix of the fully cracked material in Eq. 8, i.e.,

$$\underset{\sim}{D}^{fr} = \lim_{\mu \to 0} C^{-1}(\mu) \tag{10}$$

This relation (theorem) has been proven mathematically [6] writing the general inverse of Eq. 8 with the help of partitioned matrices. It is also physically obvious that the diagonal compliance term for a fully cracked material should approach infinity because the material has no stiffness in the direction normal to the cracks and a finite stress therefore produces an infinite strain.

The use of Eq. 9 instead of Eq. 8 in finite element programming saves programmer's effort, since only one element of the elastic compliance matrix needs to be changed. In practice one may replace C_{33} by a large number (say 10^{40}), and numerical matrix inversion then yields a matrix nearly exactly equal to Eq. 8.

The cases of an uncracked and a fully cracked material are characterized, in terms of the cracking parameter, as follows

$$\begin{aligned} &\text{uncracked:} \quad \mu = 1 \\ &\text{fully cracked:} \quad \mu = 0 \end{aligned} \tag{11}$$

Modeling of a progressive development of microcracks now obviously requires prescribing the variation of cracking parameter μ between these two limits (Eq.11). The variation of this parameter may be calibrated on the basis of a uniaxial tensile test. From tests of small samples in an extremely stiff testing machine (e.g., Evans and Marathe, 1968 [11]), it is known that the tensile stress-strain diagram exhibits a gradual decrease of strain at increasing strain (strain-softening), the softening branch being normally a few times longer than the rising branch.* Although this tensile stress-strain relation appears to be smoothly curved, it may be approximated by a bilinear stress-strain relation (Fig. 3). In such a case the cracking parameter is defined as

$$\mu^{-1} = \frac{-C_{33}^{t}}{C_{33}} \frac{\varepsilon_z}{\varepsilon_0 - \varepsilon_z} \tag{12}$$

*There is certainly a similarity between the present stress-strain relation and the stress-displacement relation used in other works [1, 14, 16, 17, 18, 27].

in which C_{33}^{t} is the downward slope of the softening portion of the tensile stress-strain diagram (negative), and ε_0 is the tensile strain in uniaxial tensile tests at which the stress is reduced to 0, i.e., continuous cracks form.

Eq. 12 for the bilinear tensile stress-strain diagram may be easily replaced by a more complicated equation corresponding to a curved tensile stress-strain diagram. However, it appears that the bilinear stress-strain diagram is sufficient to obtain a satisfactory agreement with existing test data on fracture. Moreover, the use of a straight line strain-softening diagram avoids uncertainty with the point at which unstable strain localization leading to fracture appears.

The foregoing stress-strain relations involve only the normal stress and strain components. They are obviously applicable only when the principal stress directions do not rotate during the formation of fracture. Even then, if the loading subsequent to fracture formation produces shear strains on the crack planes, the stress-strain relation must be expanded by the inclusion of shear terms (leading to a 6 x 6 stiffness or compliance matrix).

Another adjustment of the foregoing stress-strain relation is appropriate when there are significant compressive normal stresses in the directions parallel to the crack plane. In such a case it is necessary to decrease the value of the peak stress f_t' as a function of σ_x and σ_y (see [6]).

CRACK BAND MODEL FOR FRACTURE

The stress-strain relations which describe progressive microcracking may be used for modeling fracture of heterogeneous brittle materials, characterized by the existence of a microcrack zone (the fracture process zone) in front of the major crack. In materials such as rocks, concrete, certain ceramics and other, the propagation of a major crack must be always preceded by the propagation of the microcrack zone at the front of the crack. Thus, the question of propagation of the crack may be always reduced to the question of propagation of the crack band.

An essential aspect in modeling fracture by strain-softening stress-strain relations is to realize that the width of the crack band front must be considered as a material property. There are very definite theoretical reasons for this.

1. If the crack band width is considered as a variable, and the material is treated as a continuum, then an analysis of the strain-localization instability indicates that the crack band would always localize to a sharp front of vanishing width. This situation is however equivalent to linear fracture mechanics and cannot describe the test data available for rock and concrete.

2. It is meaningless to consider the crack band front to be less than a few times the size of the heterogeneties in the material, such as grain size in rock (or aggregate size in concrete). On such small

dimensions one does not have a continuum, and the foregoing continuum model would be inapplicable.

3. Finally, the mathematical model must satisfy the conditions of objectivity, which includes the requirement that the results of analysis must be independent of analyst's choice of the mesh (except for an inevitable numerical error which vanishes as the mesh is refined). It has been verified numerically, that if the width of the finite element over which the crack band is assumed to be distributed is varied, the fracture predictions vary too. The only way to achieve consistent results is to assume that the width of the crack band at the fracture front is fixed [6].

Modeling of fracture by means of stress-strain relations thus leads to a model characterized by three material parameters: the uniaxial tensile strength, f_t', the fracture energy G_f (defined as the energy consumed by crack formations per unit area of the crack plane), and the width w_c of the crack band front. Since

$$G_f = W_f \, w_c \tag{13}$$

where W_f is the work of the tensile stress per unit volume, equal to the area under the tensile stress-strain diagram (Fig. 3), the softening slope C_{33}^t is not an independent parameter and follows from the values of f_t', G_f, and w_c.

The foregoing simple stress-strain relation for tensile strain-softening has been implemented in a finite element program. For this purpose, the stress-strain relation is differentiated and used in an incremental loading analysis.

This finite element program has been used to fit various test data available in the literature. First the model was applied, with considerable success, to the fracture data for concrete [6] for which rather extensive information exists. Subsequently, the finite element model was applied to analyze fracture data for rock (private communication by B. H. Oh, Northwestern University, 1982). Excellent fits have been obtained in this manner for the maximum load data from the fracture tests for Indiana limestone, Westerly granite, Carrara marble, and Colorado oil shale [7, 8, 15, 23, 24, 25]. Two of the comparisons with tests data obtained in this manner are examplified in Fig. 4a, b. The optimum fits obtained with the present nonlinear fracture theory are indicated by the solid lines, and the optimum fits possible with linear fracture mechanics are indicated by the dashed lines. We see that the present nonlinear theory achieves a significant improvement. The same improvement was found in comparisons with other test data (B. H. Oh, 1982).

The present theory is also capable of representing the so-called resistance curves* (R-curves), i.e., the plots of apparent fracture energy vs. the length of crack extension from a notch. An example of a comparison between the present nonlinear fracture theory and the experimental R-curve measured by Hoagland at al. [15] is shown in Fig. 5.

From the analysis of test data for various rocks it appeared that the width of the crack band front for the optimum fits is approximately equal 5-times the size of the grain in rock. In the previous study of experimental data on concrete fracture, the widths of the crack band front was found to be, for the optimum fits, about 3-times the maximum aggregate size. For concrete, the set of available fracture data is larger than it is for rock, and the test results are more consistent, obviously because the differences from one concrete to another are not as large as from one rock to another. A statistical regression analysis of the test data for concrete, considered in terms of the plot of theoretical maximum load vs. the measured maximum load, both normalized with respect to the maximum load predicted by strength theory, indicated a coefficient of variation of only 6%. This is to be compared with a coefficient of variation of 24% which was found for the optimum fits by linear fracture mechanics. This statistical analysis [6] involved test results for twenty-two different concretes. A similar statistical regression analysis has been carried out also for the available data on rock fracture in which significant deviations from linear fracture mechanics are found. For these, the coefficient of variation for the predictions from the present theory was about 10% while for linear fracture mechanics predictions it was about 45%.

STRESS-STRAIN RELATIONS WHEN PRINCIPAL STRESSES ROTATE

The analyses of the fracture test data we just discussed do not necessitate general stress-strain relations because the tests are arranged so that the principal stress directions remain constant. It seems that for simulation of fracture one can often assume a model valid for constant principal stress directions even under a general loading history. This is because the rotation of principal stress directions is usually small during the relatively small loading increment in which the fracture band traverses a fixed point. The same assumption is in fact implied in the use of the J-integral, which is exact only for a nonlinear elastic material, i.e., a material described by a unique relation between stresses and strain. A similar assumption is implied in the foregoing analysis, which is also based on the use of a total stress-strain relation.

The limitation to constant principal stress directions would however be unacceptable for sharply nonproportional loading paths, which are most often encountered under dynamic loading. The question then is how can one describe progressive microcracking when the principal stress producing the microcracks rotates before complete cracks are produced. In formulating the stress-strain relation one must then observe the conditions of tensorial invariance. After considering many possibilities, it has been found that the stress-strain relation for progressive microcracking, as given in the preceding, may be generalized in the following form

*To obtain the resistance curves, the energy release must be evaluated; see [3, 5].

$$\varepsilon_{ij} = C^{sec}_{ijkm}\,\sigma_{km} \tag{14}$$

in which

$$C^{sec}_{ijkm} = C^{e\ell}_{ijkm} + b\,\varepsilon_{ik}\,\varepsilon_{jm} \tag{15}$$

Here subscripts i, j, k, m refer to cartesian coordinates x_i (i = 1,2,3), C^{sec}_{ijkm} is the tensor of secant compliances, $C^{e\ell}_{ijkm}$ is a tensor of elastic compliances, and b is a scalar parameter which may depend on the invariants of stress and strain.

An interesting aspect of Eq. 15 is that in the quadratic term, subscripts i, j and k, m do not stay together but are staggered. I may be checked that any formulation based on inelastic potentials would have to lead to terms of the type $\varepsilon_{ij}\ \varepsilon_{km}$ in which subscripts i and j stay together, and so do the subscripts k and m. As a consequence of this observation, it seems that the stress-strain relations for progressive microcracking leading all the way to complete failure cannot be formulated on the basis of inelastic potentials and normality rules.

LIMITATIONS OF INCREMENTAL ORTHOTROPIC MODELS

In the context of the preceding analysis it may be appropriate to point out certain important limitations of the incremental orthotropic stress-strain relations which have become quite popular in recent years. We now have in mind not only tensile nonlinear behavior, but general inelastic behavior including compressive and shear states.

In similarity to the relation between total stresses and total strains which was implied in our preceding modeling of progressive microcracking, the incremental orthotropic models are also applicable, in a rigorous manner, only if the principal stress directions do not rotate. However, when the application of these models is extended to rotating principal stress directions, the question does not consist merely in the path independence of the material. The orthotropic models, i.e., the relations between the increments of stresses and the increments of strain characterized by an orthotropic stiffness matrix, can in general be applied in two different ways:

1. Either the coordinate axes to which the material orthotropic properties are referred are kept fixed with regard to the material during the deformation process (although arbitrary coordinates may of course be introduced before deformation begins);

2. Or the axes of orthotropy are rotated during the deformation process (from one loading step to the next) so that they remain parallel to the principal stress directions at the beginning of each load step.

Consider the second method, which is probably used most often. The nonlinearity of the material response is due to formation of certain defects (e.g., microcracks as we consider them here, or slips) by the previous deformation history. The rotation of the axes of orthotropy means that we rotate these defects against the material. This is of course physically impossible, and would be acceptable only if the defects were either caused solely by the current stress state (and not by the previous deformations) or if they had no oriented character (e.g. round voids rather than oriented microcracks). Such an assumption is obviously unacceptable for geomaterials. In continuum mechanics of inelastic behavior it is a generally accepted fundamental fact that the material axis must remain fixed with regard to the material after the deformation begins. The symmetry properties, such as isotropy, are characterized by the fact that any coordinate axes may be chosen before the deformation begins, and the obtained responses must be equivalent. This however does not permit rotating the coordinate axes after the material has already been deformed.

Consider now the first method in which the material properties are described in terms of coordinate axes that are attached to the material as it deforms, and consider the constitutive relations

$$d\sigma_{ij} = D_{ijkm}(\underset{\sim}{\sigma})\,d\varepsilon_{km} \tag{16}$$

in which the tangential moduli tensor D_{ijkm} is a function of the stress tensor $\underset{\sim}{\sigma}$. For isotropic (precisely, initially isotropic) materials, it must be possible to apply these relations for arbitrarily chosen coordinate axes and get equivalent results. Therefore, these incremental constitutive relations must be form-invariant under any transformation on the coordinate axes. This form-invariance condition is expressed mathematically in the form (see e.g. Malvern [20]):

$$D_{pqrs}(\underset{\sim}{\sigma}') = a_{pi}\,a_{qj}\,a_{rk}\,a_{sm}\,D_{ijkm}(\underset{\sim}{\sigma}) \tag{17}$$

in which $a_{ij} = \cos(x'_i, x_j)$ = the transformation tensor containing the directional cosines of the rotated axes x'_i with respect to the original axes x_j, and $\underset{\sim}{\sigma}'$ is the stress tensor in the rotated system of coordinate axes x'_i, obtained by the tensorial transformation of stress tensor $\underset{\sim}{\sigma}$ in the original axes x_i; $\sigma'_{uv} = a_{ug}\,a_{vh}\,\sigma_{gh}$. It can be demonstrated [3] that the foregoing condition is always violated by the orthotropic incremental elastic models when they are applied using the first method. It can be also shown [3] that the lack of objectivity due to the lack of tensorial invariance can cause serious discrepancies, i.e., the predictions obtained when working with different coordinate systems can greatly differ, sometimes as much as 50%. (Note that the fact that after coordinate transformation the orthotropic matrix yields a matrix with non-zero coefficients relating normal strains and shear stresses is immaterial; this is because the orthotropic symmetry properties do not change with the coordinate transformation and apply after transformation with respect to rotated axes.)

The use of a relation between total stresses and total strains does not suffer with the limitations with the orthotropic models. In fact, differentiations of such a stress-strain relation yields an incremental stiffness or compliance matrix which is, in general, fully populated, and is not of orthotropic form (with zeros in the elements connecting shear strains and normal stresses). Neither is the incremental orthotropic matrix obtained with constitutive models based on normality rule and loading surfaces. Even the incremental stiffness matrix for the classical von Mises plastic-hardening material is not of an orthotropic form.

LIMITATIONS OF CUBIC TRIAXIAL TESTS

The recent popularity of orthotropic stress-strain relations has no doubt been caused by the recent exaggerated emphasis on the cubic triaxial tests, as opposed to the classical cylindrical triaxial tests. In cubic specimens, the principal stress axes cannot be made to rotate during the loading process, and by virtue of symmetry the principal directions of stress and strain are forced to coincide. The cubic triaxial tests have of course the advantage that they can reveal the effect of the medium principal stress. In many situations it seems to be however less important than the knowledge of the effect of rotation of principal stress directions and the effect of non-coincidence of the principal stress and strain directions (the lack of coaxiality).

To be able to observe the effect of rotating principal stress directions, other types of tests will be needed. One such type of test is a classical test in which a hollow cylinder is subjected to axial load, lateral external and internal pressure, and torsion. In this test one can induce any combination of principal stresses, and moreover one can make the principal stress directions have an angle with the specimen axis and rotate, either continuously or abruptly, during the loading process. Much more attention should be given to this type of test in the future.

CONCLUSIONS

1. Friction on microcracks is an important aspect of the deformation of geomaterials. Certain types of incremental stress-strain relations can guarantee material stability even if the normality rule is violated, due to friction.

2. Fracture of geomaterials may be modeled in terms of stress-strain relations for progressive microcracking taking place in the crack band at the front of fracture. A simple yet realistic formulation of this behavior is possible by modifying one diagonal term of the compliance matrix as a function of the extent of cracking.

3. The resulting fracture model is characterized by three parameters, the tensile strength, the fracture energy, and the width of the crack band front. This model can describe existing test data for rocks as well as concrete, both the maximum load data and the resistance curves.

4. For general loading, the tensorial invariance of the stress-strain relations under rotating principal stress directions is an important consideration. An extension of the fracture model presented is possible in terms of a total stress-strain relation.

5. The recently popular incremental orthotropic models have certain severe limitations consisting in violation of certain basic principles of continuum mechanics — either the tensorial invariance condition or the requirement that the coordinate axis used in material description may not be rotated against the material during the deformation process.

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