Prediction of breakage-induced couplings in unsaturated granular soils

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This note addresses the interplay between particle breakage, water-retention curve and deformation processes. For this purpose, a continuum theory for unsaturated granular soils is used to capture the effects of loading and/or wetting in the grain-crushing regime. The theory provides a relation to express the air-entry value of the retention curve as a function of the degree of particle breakage, thus generating a coupled hydro-mechanical formulation. These features are exploited by deriving the incremental relations that govern the hydro-mechanical response. It is then shown that breakage-dependent retention curves generate various coupling effects, such as the evolution of the degree of saturation along constant-suction paths and the pressure dependence of the wetting-collapse strains.

Since these capabilities are an outcome of coupling terms derived from grain-scale considerations, these results suggest that microstructure-inspired models are important tools to capture the feedbacks between the macro-scale response and the evolution of micro-scale attributes, as well as to minimise the recourse to phenomenological assumptions.

KEYWORDS: constitutive relations; partial saturation; particle crushing/crushability; sands; suction

INTRODUCTION

Recent studies have shown that changes of the hydraulic state are a source of crushing and compaction in coarse-grained soils subjected to high pressures (Oldecop & Alonso, 2003; Alonso et al., 2005; Ovalle et al., 2013). Similarly, new evidence has shown that crushing may alter the soil water retention curve (SWRC) of particulate media (Jamei et al., 2011). Such mechanisms are a challenge for constitutive models, as they suggest that the SWRC of crushable soils must co-evolve with the grain size distribution (GSD), making coupled hydro-mechanical formulations necessary to simulate their response. Although these couplings are controlled by specific microscopic mechanisms (i.e. grain breakage), it is arguable that the mathematical structure of models for unsaturated crushable soils must share similar features with those typically used for fine-grained materials, in which such macroscopic couplings are often observed.

It is therefore readily apparent that the formulation of models for unsaturated crushable soils is a challenging proving ground, as well as a tremendous opportunity to develop a better understanding of the microscopic origin of hydro-mechanical couplings. The purpose of this note is to tackle such challenges by using the microstructure-inspired model formulated by Buscarnera & Einav (2012), in which the co-evolution of GSD and capillary solid–fluid interactions is captured by way of statistical homogenisation. This model relies on a thermo-mechanical formulation in which the stored energy is assumed to be given by the sum of a strain-dependent term and a saturation-dependent term, in which the rich set of hydro-mechanical couplings that may emerge in crushable soils as a result of the grain-size dependence of their water-retention properties. On the other, the aim is to comment on some interesting mathematical features of the model that derive from the incorporation of microstructural variables. Indeed, some of the peculiar features displayed by this specific microstructure-inspired formulation have the potential to inspire new strategies to model hydro-mechanical coupling also in other classes of unsaturated soils.

COUPLING BETWEEN SWRC AND PARTICLE FRAGMENTATION

Comminution alters the GSD of a granular soil, affecting its capacity to retain water. Several empirical or semi-empirical models available in the literature recognise the link between GSD and SWRC (Arya & Paris, 1981; Aubertin et al., 2003; Likos & Jaafar, 2013). One of the most widely used models of this kind has been proposed by Arya & Paris (1981) (hereafter referred to as the AP model), who have used the GSD and the void ratio to estimate the water retention curve of a granular soil. Although none of these models is combined with a strategy to assess how physical and mechanical agents alter the particle grading, they all imply that any alteration of the GSD impacts the SWRC.

The unsaturated breakage model (UBM) (Buscarnera & Einav, 2012) shares similar features with the above-mentioned approaches, in that it predicts the evolution of the SWRC as a function of the alterations of the GSD induced by comminution. This model relies on a thermo-mechanical formulation in which the stored energy is assumed to be given by the sum of a strain-dependent term and a saturation-dependent term, both derived from microscopic scaling laws through statistical homogenisation. The total Helmholtz free energy potential of this model can be expressed as follows

$$
\Psi(\varepsilon^s, S, B) = (1 - \delta_M B) \psi^M(\varepsilon^s) + (1 + \delta_H B) \psi^H(S) \quad (1)
$$

where $\delta_M$ and $\delta_H$ are grading indices; $\Psi$ and $\psi_i$ are the total and reference Helmholtz free energy functions, respectively; the superscripts and subscripts M and H indicate mechanical and hydraulic components, respectively; $\varepsilon^s$ is the elastic strain, $S$ is the degree of saturation; and $0 \leq B \leq 1$ is the breakage variable quantifying the degree of comminution.
Within such formalism, the SWRC can be expressed as follows

\[
ns = - \frac{\partial W}{\partial \xi} = -(1 + \beta_B) \frac{\partial \psi^M_i(S_i)}{\partial S_i}
\]  

(2)

where \( n \) is the porosity and \( ns \) the smeared suction. The SWRC in equation (2) is factorised by a term \((1 + \beta_B)\) that evolves with the GSD through \( B \). As a result, it suggests that the evolution of the SWRC is inherently coupled with the mechanical processes. In particular, as breakage takes place, the SWRC shifts towards larger values of suction. The magnitude of such movement is controlled by the grading index \( \beta_B \), which is computed on the basis of an inverse grain-size scaling inspired by the capillary theory.

A schematic description of such alterations of the SWRC is illustrated in Fig. 1 with reference to AP and UBM models (where, for the latter, a simple hyperbolic SWRC is illustrated in Fig. 1 with reference to AP and UBM models, in which the shape of the SWRC and its cumulative GSD, \( D \), and the air-entry point of the SWRC of a soil. This model. Grain crushing is therefore likely to alter both the shape and the air-entry point of the SWRC. This aspect is confirmed by recent experiments on crushable granular materials done by Jamei et al. (2011), as well as by empirical models, in which the shape of the SWRC and its AEV depend on both the particle diameter at 10% of the cumulative GSD, \( D_{10} \), and the coefficient of uniformity, \( C_u \) (e.g. Aubertin et al., 2003).

Assessing which of the two effects is more important in a crushable soil would require an extensive set of experiments, and it is therefore beyond the scope of this note. Nevertheless, in either of the above-mentioned scenarios grain crushing causes significant changes of the water-retention capacity, which are inherently coupled with the stress–strain response. As a result, given the unique ability of the UBM model to capture the interplay between microstructure evolution, mechanical behaviour and hydrologic properties, here its simplifying assumptions are accepted, with the purpose of studying the hydro-mechanical implications of a breakage-dependent SWRC.

**INCREMENTAL CONSTITUTIVE FORMULATION**

Following Buscarnera & Einav (2012), the yield condition in the presence of coupled breakage–frictional dissipation is

\[
y(E_B, B, \sigma') = \frac{E_B}{E_c}(1 - B)^2 + \left( \frac{q}{\bar{M}p} \right)^2 - 1 \leq 0
\]  

(3)

where \( E_B = -\frac{\partial W}{\partial B} \) is the breakage energy; \( E_c \) is the critical breakage energy at the onset of comminution; \( \sigma' = \sigma - u_B \bar{\delta} + s_S \delta \) is the average skeleton stress, where \( u_B \) is the air pressure and \( \bar{\delta} \) is the Kronecker’s delta; \( \bar{\delta}, q \) are the mean and deviatoric stresses; and \( M \) is the stress ratio at frictional failure. Thus, the consistency condition can be written as

\[
d\gamma = \frac{\partial \gamma}{\partial B} dB + \frac{\partial \gamma}{\partial E_B} dE_B + \frac{\partial \gamma}{\partial \sigma'} : d\sigma' = 0
\]  

(4)

Using hyperelastic relations, the increment of the variables \( E_B \) and \( \sigma' \) can be derived

\[
d\sigma' = d \left( \frac{\partial W}{\partial \varepsilon^a} \right) = (1 - \beta_M B) \frac{\partial^2 \psi^M}{\partial \varepsilon^a \partial \varepsilon^a} : d\varepsilon^a - \beta_M \frac{\partial \psi^M}{\partial \varepsilon^a} dB
\]  

(5a)

\[
dE = d \left( - \frac{\partial W}{\partial B} \right) = \beta_M \frac{\partial \psi^M}{\partial \varepsilon^a} : d\varepsilon^a - \beta_H \frac{\partial \psi^M}{\partial S_i} dS_i
\]  

(5b)

\[\text{Fig. 1. Comparison of predicted changes in SWRC from different GSD-dependent models: (a) GSDs at varying values of } B; (b) AP retention model (Arya & Paris, 1981); (c) UBM retention model (Buscarnera & Einav, 2012)\]
Using the elastic–plastic strain decomposition (de = de - de p) and the flow rules (dε' = λ(∂ε' / ∂σ') and dε = λ(∂ε'/∂σE)'), it is possible to derive the plastic multiplier, λ, for a hydro-mechanical plastic-breakage process, as follows

\[ \lambda = \frac{1}{K_{PB}} (\xi_M : d\varepsilon - \dot{\xi}_H dS_t) \]  

(6)

where

\[ K_{PB} = \frac{\partial \varepsilon'}{\partial \sigma} \xi_M - \frac{\partial \varepsilon'}{\partial \sigma} \dot{\xi}_B \]

is the plastic-breakage modulus, y' is the yield function in the dissipative space and the terms \( \xi_M, \dot{\xi}_H \) and \( \dot{\xi}_B \) are given by

\[ \xi_M = \frac{\partial \varepsilon'}{\partial \sigma} \frac{\partial \psi_M}{\partial \varepsilon'} + (1 - \tilde{\varphi}_M) \frac{\partial \varepsilon'}{\partial \sigma} \frac{\partial \psi_M^0}{\partial \varepsilon'} \]

\[ \dot{\xi}_B = \frac{\partial \varepsilon'}{\partial B} \frac{\partial \psi_M}{\partial \varepsilon'} + \dot{\xi}_H = \frac{\partial \varepsilon'}{\partial B} \frac{\partial \psi_M^0}{\partial \varepsilon'} \]  

(7)

The incremental constitutive relations can be written in matrix form as follows

\[ d\Sigma = [D' - (I_b + I^p)]dE \]  

(8)

where \( \Sigma = [\sigma' \ ns']^T \) and \( E = [\varepsilon (-S_t)]^T \) are generalised stress and strain vectors; \( D' \) is an elastic matrix, while the matrices \( I_b \) and \( I^p \) reflect the effect of breakage and plasticity, respectively

\[ D' = \begin{bmatrix} (1 - \tilde{\varphi}_M) \frac{\partial \psi_M}{\partial \varepsilon'} & 0 \\ 0 & (1 + \tilde{\varphi}_M) \frac{\partial \psi_M^0}{\partial \varepsilon'} \end{bmatrix} \]  

(9a)

\[ I_b = \frac{1}{K_{PB}} \frac{\partial \varepsilon'}{\partial B} \begin{bmatrix} \frac{\partial \psi_M}{\partial \varepsilon'} \otimes \xi_M - \frac{\partial \psi_M^0}{\partial \varepsilon'} \otimes \dot{\xi}_H \\ \dot{\xi}_H \end{bmatrix} \]  

(9b)

\[ I^p = \frac{1}{K_{PB}} \frac{\partial \varepsilon'}{\partial B} \begin{bmatrix} \frac{\partial \psi_M}{\partial \varepsilon'} \otimes \xi_M - \frac{\partial \psi_M^0}{\partial \varepsilon'} \otimes \dot{\xi}_H \\ 0 \end{bmatrix} \]  

(9c)

where the symbols 0 indicate zero matrices.

Equation (8) shows that the constitutive response is the sum of an elastic contribution and two inelastic terms. It can be noticed that, while the mechanical and hydraulic properties are decoupled in the elastic regime (\( D' \) in equation (9a)), cross-coupling terms are introduced by the comminution matrix \( I_b \). These terms depend on the evolution of the breakage variable \( B \), and can therefore be interpreted as a direct outcome of the evolving microstructure. Similarly, the onset of plastic strains introduces a further off-diagonal term in the constitutive equations through \( I^p \), which is also inherently coupled with the growth of \( B \) through the flow rules. The constitutive properties of such particular coupled formulation will be explored hereafter by simulating the effects of loading and/or wetting paths in the crushing regime.

**HYDRO-MECHANICAL COUPLING IN CRUSHABLE SOILS**

The constitutive relations can be defined by assigning specific expressions to the potentials in equation (1). In the following, a hyperbolic retention curve and a pressure-dependent elastic potential have been used, calibrating their parameters on the basis of available data for Toyoura sand (Fig. 2; Table 1). Given the simplicity of the selected

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**Table 1. Model parameters for Toyoura sand**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\varphi}_M )</td>
<td>0.757</td>
</tr>
<tr>
<td>( \tilde{\varphi}_H )</td>
<td>0.25</td>
</tr>
<tr>
<td>( K )</td>
<td>5500</td>
</tr>
<tr>
<td>( G )</td>
<td>5022</td>
</tr>
<tr>
<td>( M )</td>
<td>1.2</td>
</tr>
<tr>
<td>( \omega )</td>
<td>64</td>
</tr>
<tr>
<td>( E_c )</td>
<td>21.9 MPa</td>
</tr>
<tr>
<td>( K_c )</td>
<td>4 kPa</td>
</tr>
</tbody>
</table>
retention model, the simulations can only approximate the sharp kink of the data at the AEV of the sand. This problem, however, can be removed by using more sophisticated retention functions. Further details about the implications of different elastic potentials and retention models can be found in Zhang & Buscarnera (2014).

Several oedometric tests have been simulated at different values of initial suction (Fig. 3(a)). The model predicts that the yielding stress associated with the maximum rate of comminution (and, hence, with the maximum rate of compaction) tends to increase with suction, that is, when drier conditions are approached. The breakage evolution curves eventually converge at high stress levels, suggesting the effect of the initial hydraulic state becomes less important at advanced stages of crushing. During the early stages of loading, the compressibility of unsaturated samples is predicted to decrease compared to saturated conditions, with larger values of suction magnifying such a stiffening effect. By contrast, unsaturated states are predicted to be characterised by larger values of compressibility when loaded in the high-pressure regime, where dry states involve a larger rate of compaction compared to saturated states. Fig. 3(b) illustrates the effect of wetting paths, using the two simulations at \( s = 0 \) and \( s = 1 \) MPa as a reference (marked as tests a and b, respectively). The tests c, d, e and f are simulated wetting paths imposed at different values of vertical net stress. After suction is decreased to zero, all the simulated tests exhibit a compression–breakage response similar to that associated with the saturated case (test a).

Figure 4(a) presents the simulated stress paths for tests a \((s = 0)\) and b \((s = 1 \text{ MPa})\). It is readily apparent that the predicted expansion of the elastic domain under unsaturated conditions (test b) produces a steeper stress path at low stress levels compared to the saturated one (test a). Nevertheless, once a considerable amount of breakage has occurred (e.g. when \( B = 0.55 \)) the two paths tend to converge. These effects are a direct consequence of the assumed shape of the hydraulic potential, which influences the predicted changes in size and shape of the elastic domain (Buscarnera & Einav, 2012). Such changes of the yield surface are the main cause of the stiffer compression response in Fig. 3(a), which is eventually suppressed in the post-yielding regime when the two simulated stress paths converge. Fig. 4(b) illustrates the evolution of SWRC and GSD at selected states for saturated conditions (path with square symbols) and unsaturated conditions (path with circular symbols) and unsaturated conditions (path with square symbols).

![Fig. 3. Predicted compression response and breakage evolution for: (a) simulated suction controlled oedometric compression at different levels of suction; (b) simulated suction controlled wetting stages at different values of vertical net stress](image-url)

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Fig. 4. (a) Stress paths during simulated oedometric compression (dotted lines indicate yield surfaces at selected states for saturated conditions (path with circle symbols) and unsaturated conditions (path with square symbols); (b) evolving hydraulic state and SWRCs for the simulated test b
CONCLUSIONS

The authors have used the breakage mechanics framework for unsaturated granular media to investigate the coupled hydro-mechanical behaviour of crushable soils, giving emphasis to the dependence of the water-retention curve on the degree of particle breakage. A notable feature of the selected approach is that the coupled effects have not been assumed a priori, but have rather been deduced from micromechanical considerations based on the capillary theory and a procedure of statistical homogenisation.

To study the implications of a water-retention curve that evolves during comminution, the incremental hydro-mechanical relations of the model have been derived. Such derivation has shown that particle breakage introduces cross-coupling terms able to link the hydrologic response and the compression behaviour of a crushable granular soil. Using this formulation, a series of oedometric tests have been simulated, showing that breakage-dependent functions enable the simulation of a broad range of coupled effects, such as suction-dependent stress thresholds at the onset of comminution, stiffer compression response for increasing values of suction and wetting-induced breakage. The simulated breakage events, either caused by loading or wetting, are always predicted to be accompanied by a simultaneous increase in the suction air-entry point. Furthermore, it has been shown that the mathematical structure of the model provides the possibility to reproduce the pressure-dependence of the wetting-induced strains, thus capturing the existence of a stress level associated with the maximum collapse.

Such hydro-mechanical feedbacks have been captured by a mathematical formulation considerably different from those typically used to mimic similar patterns in other classes of unsaturated soils. One key aspect that has led to this result is that hydro-mechanical coupling is not directly injected into the constitutive functions through phenomenology, but is rather predicted by using scaling relations that link the microstructural attributes (e.g. the particle size) to the continuum properties. Hence, these results suggest that microstructure-inspired arguments can lead to mathematical formulations capable of minimising the number of phenomenological assumptions required to capture complex macroscale effects. It is the authors’ opinion that this approach to constitutive modelling is susceptible to future generalisations based on the incorporation of other micro-scale attributes (e.g. pore size, inter-particle bonds), as well as on a more systematic use of micro-scale characterisation techniques.

ACKNOWLEDGEMENT

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NOTATION

- $B$ breakage index
- $c_u$ coefficient of uniformity
- $D_{10}$ particle diameter at 10% of the cumulative GSD
- $D^e$ elastic stiffness matrix
- $d$ grain diameter
- $E_B$ breakage energy
- $e$ void ratio
- $K, G$ elastic stiffness constants
- $K_{PB}$ breakage-plastic modulus
- $K_r$ retention curve parameter
- $M$ stress ratio at failure
- $n$ porosity
- $S_i$ degree of saturation
- $s$ suction
- $\lambda$, $\gamma^*\delta$ yield function in true and dissipative stress space
- $\Gamma$, $\Gamma^*\delta$ stiffness reduction terms due to breakage and plasticity
- $\epsilon$, $\epsilon^s$, $\epsilon^{sh}$ total, plastic and elastic strain tensors
- $\delta_{h2}$, $\delta_{h1}$ mechanical and hydraulic grading indices
- $\lambda$ plastic multiplier
- $\xi_{MK}$, $\xi_{SH}$ consistency coefficients
- $\Sigma$, $E$ generalised stress and strain vectors
- $\sigma^*$, $\sigma^p$, $\sigma^q$ total, plastic and elastic stress tensors
- $\alpha_i$ total vertical stress
- $\sigma_{net}$, $\sigma_{net}^*$ net stress tensor, vertical net stress
- $\Psi$ total Helmholtz free energy
- $\psi^M$, $\psi^H$ mechanical and hydraulic reference Helmholtz free energy
- $\omega$ breakage-plasticity coupling angle

REFERENCES


