Experimental characterization of efficient second harmonic generation of lamb wave modes in a nonlinear elastic isotropic plate

Kathryn H. Matlack,[1] Jin-Yeon Kim,[2] Laurence J. Jacobs,[1,2,a)] and Jianmin Qu[3]

[1]*Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0405, USA*

[2]*School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0355, USA*

[3]*Department of Civil and Environmental Engineering, Northwestern University, Evanston, Illinois 60208, USA*

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This research experimentally characterizes the efficiency of Lamb wave mode pairs to generate the cumulative second harmonic in an undamaged aluminum plate. Previous research developed the theoretical framework for the characteristics of second harmonic generation of Lamb waves in nonlinear elastic plates, and identified five mode types where the amplitude of the measured second harmonic should increase linearly with ultrasonic wave propagation distance. The current research considers one of these five mode types, Lamb wave mode pairs at the longitudinal velocity, and experimentally confirms the theoretically predicted ratios of the rate of accumulation of the second harmonic amplitude versus propagation distance for two different Lamb wave mode pairs. By comparing these rates of accumulation, these experimental results are used to characterize the measurement efficiency of the mode pairs under consideration. [doi:10.1063/1.3527959]

I. INTRODUCTION

Previous research theoretically demonstrated that specific pairs of Lamb wave modes generate a cumulative second harmonic wave when an originally monochromatic wave is launched into a nonlinear elastic isotropic plate.[1–3] This material nonlinearity is inherent in the crystalline structure of aluminum.[4] These Lamb wave mode pairs have the potential to characterize material nonlinearity, since additional research has shown that the second harmonic generation (SHG) of Lamb waves is related to material nonlinearity and fatigue damage.[5–7] In terms of measuring material nonlinearity, Lamb waves have advantages because they can: (1) propagate over long distances, as opposed to pure longitudinal or bulk waves that are more suitable for a through-the-thickness measurement; and (2) interrogate the entire depth of the material, unlike Rayleigh waves that only propagate along the surface of the material. Thus, the ability to experimentally characterize SHG of different Lamb wave modes could provide a more efficient means of measuring material nonlinearity.

Recent experimental work has investigated specific Lamb wave mode pairs exhibiting SHG, although this work was limited to one mode pair at time—a single mode pair at the longitudinal phase velocity of the plate material in Pruell *et al.*[5] and Bermes *et al.*,[7] and a mode pair at crossing points (in the dispersion curve) in Deng *et al.*[6] Previous research using longitudinal waves has shown a direct relation between the material's nonlinear parameter and SHG from an ultrasonic wave.[8,9] Other theoretical models demonstrate a direct correlation between material nonlinearity and accumulated material damage prior to crack initiation.[10,11] Research aimed at measuring material nonlinearity with Lamb waves could thus lead to improved techniques for quantitatively monitoring accumulated material damage.

The objective of the current research is to experimentally confirm the behavior theoretically predicted by Müller *et al.*[1] for the rate of accumulation of the second harmonic of different Lamb wave mode pairs in an undamaged 6061 T6 aluminum alloy plate. By comparing these rates of accumulation, these experimental results are used to characterize the measurement efficiency of the Lamb wave mode pairs under consideration.

II. SHG

Following de Lima and Hamilton[2] and Müller *et al.*,[1] consider an isotropic, homogeneous, nonlinear elastic, and infinite plate with stress-free boundary conditions at the surface. A nonlinear equation of motion is developed from a balance of linear momentum, the second-order constitutive law, and the Lagrangian strain tensor

$$(\lambda + 2\mu)\,\nabla\,(\nabla \cdot \boldsymbol{u}) - \mu\,\nabla\, \times (\nabla \times \boldsymbol{u}) + \nabla \cdot \bar{\boldsymbol{S}} = \rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \quad (1)$$

where $\boldsymbol{u}$ is the displacement vector, t is time, $\bar{\boldsymbol{S}}$ is the nonlinear portion of the second Piola–Kirchhoff stress, and λ and μ are Lamé's constants. The solution is formulated using a perturbation approach, which assumes the total displacement field can be expressed as the sum of the primary wave (at frequency ω) and a secondary wave (the second harmonic at frequency 2ω); this perturbation solution is possible since the amplitude of the second harmonic is much smaller than that of the primary wave. A full solution for the primary

[a)]Electronic mail: laurence.jacobs@coe.gatech.edu.

wave is given in Graff.[12] The secondary wave solution is developed with a modal expansion technique.[1,2]

Specific conditions must be satisfied for SHG. The first condition, referred to as *phase velocity matching*, requires equal phase velocities of the primary and the secondary waves. A second condition, referred to as *group velocity matching*, requires equal group velocities of the primary and secondary waves, and a third condition is nonzero power flux from the primary to the secondary wave. If all these conditions are satisfied, the trend of the second harmonic amplitude (amplitude of the secondary wave) can be described by the parameter r_s, defined as rate of accumulation of the second harmonic,

$$\frac{A_2}{A_1^2} = z r_s, \tag{2}$$

where A_1 and A_2 are the measured amplitudes of the primary and secondary waves, respectively. The value of r_s depends on the specific Lamb wave mode pair under consideration, and includes other amplitude terms from the secondary wave besides the propagation distance, z and the primary wave amplitude, A_1, squared. This r_s parameter accounts for the different rates of energy transfer between the primary and secondary waves, lumping together all the information about the second harmonic amplitude that should remain constant for a specific Lamb wave mode pair. So r_s is similar to the relative acoustic nonlinear parameter, β', defined in previous work, and is used in this study to quantify the measurement effects that will cause differences between the theoretical and experimental values. Note that β' and the absolute material nonlinear parameter, β, have been experimentally correlated in previous research.[5,7]

Two Lamb wave mode pairs with phase velocity equal to the longitudinal velocity (labeled L_1 and L_2 in Fig. 1) are selected for investigation due to ease of generation and detection—both of the primary waves in these mode pairs have the fastest group velocity at their respective frequencies, so they are well separated from other Lamb modes. Another advantage with these mode pairs is their higher rates of accumulation, e.g., 3.76 for the s1-s2 mode versus 2.49 for the a2-s4 mode pair. A second type of Lamb wave mode pair, (labeled C_1 in Fig. 1) the crossing-points mode, is difficult to generate and detect with the current experimental procedure since the other modes at these crossing points are generated much more efficiently. The critical parameters of the two selected Lamb wave mode pairs at the longitudinal velocity, labeled L_1 and L_2, and one mode pair at crossing points, labeled C_1 (presented for comparison purposes), are given in Table I. A comparison of the theoretical values for the rate of accumulation of the second harmonic as computed in Müller *et al.*[1] is shown in Fig. 2.

III. METHOD

A. Experimental procedure

Lamb wave modes are excited using wedge generation[5,7] in an undamaged 6061 T6 aluminum plate with thickness of 1.6 mm. The wedge generation method is modified by coupling the wedge to the aluminum plate with salol (phenyl

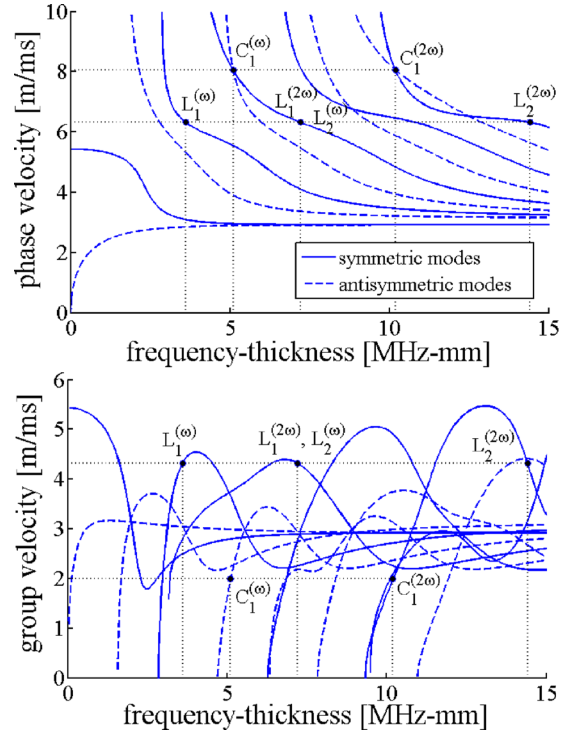

FIG. 1. (Color online) Symmetric and antisymmetric Lamb wave modes in terms of (a) phase velocity and (b) group velocity, showing experimental mode pairs (superscripts on mode pair designations distinguish between primary and secondary mode).

salicylate), producing a consistent solid-state coupling (as opposed to liquid coupling) between the wedge and plate to more efficiently generate modes with only in-plane displacement at the surface[13] [i.e., $u_y(h)=0$ and $u_z(h)\neq 0$]. A liquid coupling—a thin film of oil—is chosen for the receiving wedge since the oil coupling produces less variability than the solid coupling. There is a tradeoff between variability and strength of second harmonic signal in the different types of coupling. With the fluid coupling, it is possible to detect a small amount of second harmonic displacement with much less variability. The experimental setup produces an approximate phase and group velocity matching of modes (since it is difficult to physically excite a mode at a single particular frequency with the transducer setup), and this slight deviation excites approximate matching mode pairs that have a nonzero out-of-plane displacement at the surface. On the other hand, the solid coupling can detect the dominant in-plane displacements but with increasing variability. The reduction in variability is chosen over strength of signal to more clearly distinguish the trend in the measured acoustic nonlinear parameter. As shown in the experimental results, the oil coupling can detect the increasing second harmonic amplitude up to a finite distance.

A high-power-gated amplifier (RITEC RAM-5000 Mark IV) generates the input electrical signal of frequency either 2.25 MHz or 4.5 MHz (for the s1-s2 mode pair and s2-s4 mode pair, respectively) with 35 cycles for sufficient acoustic energy. The generated signal is at a voltage of 90% of the maximum power of the RITEC amplifier ($\sim$734 V_{pp} with transducer loading). This signal is fed into a narrowband transducer (either Panametrics X-1055 or X-1056, depending

TABLE I. Summary of Lamb wave mode pair parameters. The normalized displacement of the primary mode at the surface of the plate, h, is given in terms of in-plane displacement, $\bar{u}_z(h)$, and out-of-plane (normal) displacement, $\bar{u}_y(h)$ (these displacements are normalized by the displacement of the primary wave in mode pair s1-s2; displacements are calculated with the software DISPERSE).

Mode pair	Ref.	f_d (MHz mm)	c_{ph} (m/s)	c_g (m/s)	$\bar{u}_y(h)$	$\bar{u}_z(h)$	r_s
s1-s2	L_1	3.603	6320	4326	0	1.000	3.76
s2-s4	L_2	7.206	6320	4326	0	0.499	15.05
a2-s4	C_1	5.095	8057	2000	0	0.705	2.49

on the mode pair) with center frequency of either 2.25 or 5 MHz of radius 6.25 mm and received by a narrowband transducer (either Panametrics A-109 or A-111) with center frequency of either 5 or 10 MHz. The angle of the wedge is designed to excite the primary wave—specifically, the angle depends on the wedge material and the phase velocity of the Lamb wave mode to be generated. The receiving transducer simultaneously detects the primary and secondary wave amplitudes ($\bar{A}_1$ and $\bar{A}_2$, respectively). The center frequency of the receiver matches the frequency of the secondary wave (2ω), thus increasing the signal-to-noise ratio (SNR) while still detecting the primary wave at a frequency of ω; the amplitude of the primary wave is inherently much larger than the second harmonic (amplitude of the secondary wave). The measured time-domain signal is transferred to an oscilloscope, averaged 1000 times to further improve the SNR, and then transferred to a PC for post digital processing. All measurements are taken in the far-field, at propagation distances of 20 to 50 cm at increments of 2.5 cm. Each measurement is repeated three times, and the receiving wedge is completely removed and reattached between each measurement (note that the transmitting wedge remained attached to the plate throughout the entire measurement set to reduce variability).

B. Signal processing

A crucial issue with nonlinear Lamb wave measurements is how to accurately extract the amplitudes of the primary and secondary waves from an experimentally measured time-domain signal. Lamb waves are dispersive and multimodal, and it is difficult to experimentally excite only one mode. This research uses a time-frequency representation,[5,7] the short-time Fourier transform (STFT). For example, representative time slices at the first (ω) and second harmonic (2ω) frequencies of a typical measurement that show the amplitudes of the primary and secondary waves, $\bar{A}_1$ and $\bar{A}_2$, is provided in Fig. 3.

The question arises as to what signal processing parameters to use in the STFT analysis, and how to determine these parameters for different Lamb wave mode pairs, not simply the Lamb wave mode pairs investigated in this analysis. The window size greatly affects the extracted amplitudes of the measured Lamb waves—a very narrow window size has greater resolution in the time domain but poor resolution in the frequency domain, and the opposite is true of a very wide window. Previous experimental work on the s1-s2 mode pair used a narrow window size, since these modes arrived first in time[5] but this is not necessarily the case with all Lamb wave mode pairs. Therefore, a more robust procedure is developed.

The measured parameter, $\bar{A}_2/\bar{A}_1^2$, is inherently dominated by the primary wave amplitude (since this term is squared), so it is crucial that the primary wave amplitude extracted from the STFTs is not adversely influenced by the signal processing parameters. Since the primary wave amplitude is simply a linear ultrasonic wave propagating through a waveguide, the trend of its amplitude over propagation distance can be predicted with a diffraction model through a plate that accounts for geometric effects of a finite source and receiver—in this case the transducer/wedge assembly. If the primary wave amplitude follows the expected trend, it is confirmed that the primary wave amplitude is extracted correctly. To model this trend, the solution for a time harmonic point source in an arbitrary direction is modified to model the transducer/wedge assembly.[14] This solution, detailed in Achenbach and Xu,[14] decomposes the point source into horizontal and vertical components, and develops a modal expansion solution in terms of symmetric and antisymmetric

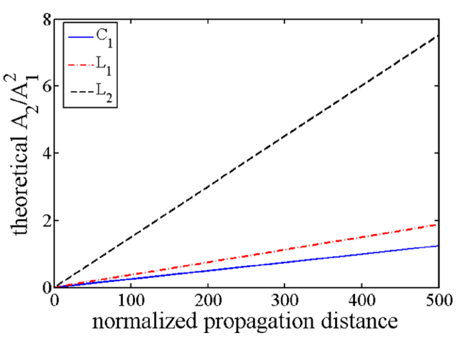

FIG. 2. (Color online) Theoretical comparison of rate of accumulation of the second harmonic of selected mode pairs.

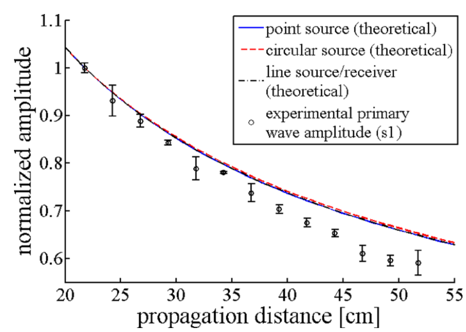

FIG. 3. (Color online) Time slices of s2-s4 received signal at a propagation distance of $z=35$ cm at the first and second harmonic frequency.

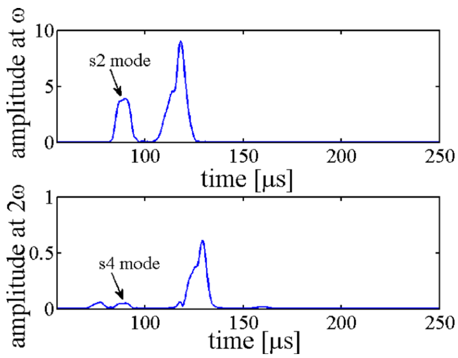

FIG. 4. (Color online) Measured primary wave amplitude of s1-s2 mode pair and the diffraction model for a point source, a circular source, and a line source/receiver.

Lamb wave modes. While numerical integration schemes make a close approximation of the experimental excitation difficult, this study shows that different geometric cases, specifically a circular source or a line source/receiver pair, have very little deviation from the point source solution (in terms of normalized amplitude) in the far field. An example of how the diffraction models, in terms of a point source, a circular source, and a line source/receiver, correlate to the experimentally measured results is given in Fig. 4.

Finally, differences in the primary wave amplitude of different mode pairs due to frequency and signal processing effects must be taken into account. The primary mode amplitude from each mode pair must be normalized to account for diffraction effects at different frequencies. Since the described procedure for determining the signal processing parameters could warrant different window sizes for different Lamb wave modes pairs, the amplitudes must be normalized to account for the window size in order to accurately compare measurements of different Lamb wave mode pairs.

C. Measurement analysis

In order to relate the measured amplitude ratio, $\bar{A}_2/\bar{A}_1^2$, to the theoretical rate of accumulation, r_s, the influence of any experimental variations must be accounted for. The amplitude ratio has the form

$$\frac{\bar{A}_2}{\bar{A}_1^2} = z\beta' - \beta_0, \qquad (3)$$

where β' is the relative nonlinear parameter and β_0 is the extraneous measurement nonlinearity inherent to the experimental setup and procedure. The relative nonlinear parameter is found by taking the slope of linear fit of the amplitude ratio over propagation distance. To isolate the rate of accumulation from measurement nonlinearity, the following normalization is used on the amplitude ratio:

$$\left(\frac{\bar{A}_2}{\bar{A}_1^2}\right)_{norm} = \left(\frac{\bar{A}_2}{\bar{A}_1^2}\right)_z - \left(\frac{\bar{A}_2}{\bar{A}_1^2}\right)_{z_0} = \beta'(z - z_0). \qquad (4)$$

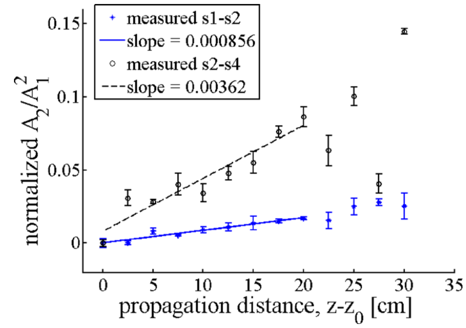

FIG. 5. (Color online) Comparison of experimental β' for s1-s2 and s2-s4 mode pairs. The ratio of the measured rate of accumulation is 4.23.

IV. RESULTS AND DISCUSSION

A. Experimental results

The measured (normalized) amplitude ratio, $(\bar{A}_2/\bar{A}_1^2)_{norm}$, over (normalized) propagation distance for both the s1-s2 mode pair and s2-s4 mode pair is shown in Fig. 5. The error bars show the measured standard deviation of the three measurement sets. The linear increase over propagation distance can be seen up to a propagation distance of 42 cm in both measurement sets, after which there is no clear trend.

The s1-s2 measurements show that two unwanted modes are generated—the a1 and a2 modes–though neither influence the primary nor secondary wave amplitudes. Out of all possible modes at the first and second harmonic frequencies (ω and 2ω), the primary and secondary wave modes have the fastest group velocity, giving sufficient modal separation beginning around 20 cm. The s1-s2 mode pair also has the advantage of having the lowest first harmonic frequency out of all the mode pairs shown in Müller *et al.*[1] to exhibit SHG. This is an advantage because fewer modes occur at lower frequencies. For example, five modes occur at the primary frequency (f_1=2.25 MHz) whereas eight modes occur at the secondary frequency (f_2=4.5 MHz).

The s2-s4 measurements show that three unwanted modes are excited—the a2, s3, and s5 modes. The s3 mode slightly influences the primary wave amplitude, since the portion of the s3 mode that is close in frequency to the primary wave propagates with the same group velocity. However the primary wave is more strongly excited than the s3 mode, so its influence is small. Note that the secondary wave, s4, does not propagate with the fastest group velocity of modes at the frequency 2ω. The secondary wave amplitude, $\bar{A}_2$, as shown in Fig. 3, is located using the theoretical time of arrival.

B. Comparison of mode pairs

The ratio between measured relative nonlinear parameters of the s2-s4 and s1-s2 mode pairs,

$$\frac{(\beta')_{s2\text{-}s4}}{(\beta')_{s1\text{-}s2}}, \qquad (5)$$

is 4.23, as shown in Fig. 5, and the theoretical ratio,

$$\frac{(r_s)_{s2\text{-}s4}}{(r_s)_{s1\text{-}s2}}, \qquad (6)$$

is 4.00, as calculated using the model in Müller *et al.*[1] It can be shown that the ratio of four comes from the dependence of the second harmonic amplitude on frequency, that is, $A_2 \propto \omega^2$. The experimental ratio takes into account differences in transducer generation efficiency (note that a different set of transducers was used for the s1-s2 measurements and the s2-s4 measurements), signal processing effects, and attenuation in the Plexiglas wedges at different frequencies while other variations such as those associated with the coupling are difficult to quantify and thus produce some error. The experimentally measured ratio is in good agreement with the theory, further confirming the suitability of both mode pairs for SHG. While the s1-s2 Lamb wave mode pair is a better choice for SHG with the current experimental technique due to excitation of fewer modes and less influence from these modes on the primary and secondary wave amplitudes, the higher rate of accumulation with the s2-s4 mode shows that an improved experimental technique with this Lamb wave mode pair could have higher SHG efficiency. Finally, it is interesting to note that these experimental results suggest that the absolute nonlinearity parameter of Lamb waves should be in the form, $\beta \propto A_2/(z\omega^2 A_1^2)$, which is exactly the same form as in the case of longitudinal[8,15,16] and Rayleigh waves.[17–19] While this form has been used for Lamb waves,[7] it has not been proven theoretically, nor has it been postulated based on experimental evidence.

V. CONCLUSION

This research experimentally investigates two Lamb wave mode pairs that exhibit SHG, the s1-s2 mode pair and the s2-s4 mode pair. It has been theoretically shown that there is a possibility of five different types of Lamb wave mode pairs that generate the cumulative second harmonic, and as of yet a comparison of feasibility and practicality of these Lamb wave mode pairs has not been reported in the literature. For each mode pair, the primary mode is generated in an aluminum plate using the wedge/transducer method and a solid coupling to efficiently excite the pure in-plane displacements at the surface. A wedge/transducer detection method simultaneously receives the primary and secondary modes at increasing propagation distances.

Both Lamb wave mode pairs show the expected linear increase in the relative nonlinear parameter with propagation distance up to sufficiently far propagation distance (42 cm). Experimental results show the ratio of the relative nonlinear parameter of the s2-s4 mode pair to that of the s1-s2 mode pair is 4.23, which is in good agreement with the theoretically predicted ratio of rates of accumulation. The s1-s2 mode pair shows fewer unwanted modes generated with no influence on the measured amplitudes, so this mode pair is preferred with the current experimental technique. However, since the s2-s4 mode pair shows an experimental rate of accumulation of the second harmonic (β') four times higher than the s1-s2 mode pair, the s2-s4 mode pair could be used with a higher efficiency with an improved experimental method.

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