On the acoustic-radiation-induced strain and stress in elastic solids with quadratic nonlinearity (L)

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This letter demonstrates that an eigenstrain is induced when a wave propagates through an elastic solid with quadratic nonlinearity. It is shown that this eigenstrain is intrinsic to the material, but the mean stress and the total mean strain are not. Instead, the mean stress and total means strain also depend on the boundary conditions, so care must be taken when using the static deformation to measure the acoustic nonlinearity parameter of a solid. © 2011 Acoustical Society of America. [DOI: 10.1121/1.3583501]

I. INTRODUCTION

When a wave propagates through an elastic solid with quadratic nonlinearity, static strain or mean strain may be generated. Such mean strain is called acoustic-radiation-induced strain. Several recent studies have used the acoustic-radiation-induced strain to measure the acoustic nonlinearity parameter.[1–6] It is well known that the acoustic-radiation-induced strain is related directly to the static term in the displacement. However, there has been some confusion in the literature regarding the magnitude of the static term in the displacement solution to the nonlinear equation.[7–10] The purposes of this note are (1) to clarify these confusions and (2) to re-interpret the concepts of acoustic-radiation-induced strain and stress.

To begin, let us consider a half-space defined by $x \geq 0$, where x is the Lagrangian (or material) coordinate describing the location of the material particle in the initial ($t = 0$) state. At any given time t, the displacement of the particle x from the initial configuration is denoted by $u(x, t)$. Deformation of the elastic body can then be described by the Lagrangian strain

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2. \tag{1}$$

We assume that the half-space is made of an elastic solid with quadratic nonlinearity, i.e., the normal (first Piola–Kirchhoff) stress is related to the Lagrangian strain/displacement gradient in the x-direction through

$$\sigma = \rho c^2 \left[\varepsilon - \frac{\beta + 1}{2}\varepsilon^2\right] = \rho c^2\left[\frac{\partial u}{\partial x} - \frac{\beta}{2}\left(\frac{\partial u}{\partial x}\right)^2\right], \tag{2}$$

where ρ is the mass density, c is the longitudinal phase velocity, and β is the acoustic nonlinearity parameter, all for the elastic solid in the undeformed (initial) state.

For isotropic elastic solids, the acoustic nonlinearity parameter is a dimensionless number given by[11]

$$\beta = -(3 + \eta), \quad \eta = \frac{2(l + 2m)}{\lambda + 2\mu}, \tag{3}$$

where λ and μ are the Lamé constants, and l and m are the Murnaghan third order elastic constants. If the material is linearly elastic, i.e., $l = m = 0$, then $\beta = -3$. In other words, $\beta = -3$ is purely due to the geometrical nonlinearity introduced in the Lagrangian finite strain [see Eq. (1)]. The material nonlinearity is represented by η.

Substituting Eq. (2) into the equation of motion $\partial \sigma / \partial x = \rho \partial^2 u / \partial t^2$ leads to the displacement equation of motion governing longitudinal wave propagation in the x-direction,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}. \tag{4}$$

We first consider the case where a harmonic displacement is prescribed on the boundary, for example,

$$u(0, t) = U \sin(\omega t), \tag{5}$$

where $\omega = kc$ is the circular frequency and k is the wavenumber. It can be easily shown by a straightforward perturbation technique that, for $|\beta U k^2 x| \ll 1$, the solution to the boundary value problem given by Eqs. (4) and (5) can be written as

$$u(x,t) = U \sin\left[\omega\left(t - \frac{x}{c}\right)\right]$$
$$+ A\beta x + \frac{\beta U^2 \omega^2}{8c^2}x\cos\left[2\omega\left(t - \frac{x}{c}\right)\right], \tag{6}$$

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where A is an arbitrary constant. Note that, unlike the corresponding linear problem where A must be zero because the solution must be bounded as $x \rightarrow \infty$, the constant A in Eq. (6) does not have to be zero because Eq. (6) is valid only for finite x.

The corresponding stress follows from substitution of Eq. (6) into Eq. (4),

$$\frac{\sigma(x,t)}{\rho c^2} = -Uk \cos\left[\omega\left(t-\frac{x}{c}\right)\right] - \beta\left(A + \frac{U^2 k^2}{4}\right)$$
$$- \frac{\beta U^2 k^2}{8} \cos\left[2\omega\left(t-\frac{x}{c}\right)\right] + \frac{\beta U^2 k^3}{4} x \cos\left[2\omega\left(t-\frac{x}{c}\right)\right].$$
(7)

In deriving Eq. (7), we have neglected terms higher ordered than $\beta U^2 k^2$

Note that Eq. (6) satisfies both the governing Eq. (4) and the boundary condition (5), at least to the order of $\beta U^2 k^2$. Yet it contains an arbitrary constant A. To overcome this non-uniqueness of the solution, most authors simply set $A = 0$. Others,[8] for example, argued, based on physical grounds, that the static part of the stress must vanish. This leads to

$$A = -\frac{U^2 k^2}{4}.$$
(8)

Another approach was taken by Cantrell,[12] who has shown that in addition to the governing equation, a forward propagating wave in quadratic nonlinear materials must also satisfy the consistency condition,

$$\frac{\partial u}{\partial t} = \frac{2c}{3\beta}\left[\left(1 - \beta\frac{\partial u}{\partial x}\right)^{3/2} - 1\right].$$
(9)

This is valid for any β and is independent of the boundary conditions. It can be easily shown that in the limit of $\beta \rightarrow 0$, the preceding reduces to the well-known equation,[13]

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}.$$
(10)

The consistency condition (9) provides an additional equation to uniquely determine the constant A. By substituting Eq. (6) into Eq. (9), we arrive at

$$A = -\frac{U^2 k^2}{8}.$$

This leads to the following solution,

$$\frac{u_D}{U} = \sin\left[\omega\left(t-\frac{x}{c}\right)\right] + \frac{\beta U k^2}{8} x\left(1 + \cos\left[2\omega\left(t-\frac{x}{c}\right)\right]\right),$$
(11)

$$\frac{\sigma_D}{\rho c^2} = -Uk \cos\left[\omega\left(t-\frac{x}{c}\right)\right] - \frac{\beta U^2 k^2}{8}\left[1 + \cos\left[2\omega\left(t-\frac{x}{c}\right)\right]\right.$$
$$\left. + 2kx \cos\left[2\omega\left(t-\frac{x}{c}\right)\right]\right].$$
(12)

In the rest of this paper, our discussions will be based on solutions that satisfy the consistency condition (9). The subscript D in the preceding expressions is to indicate that the solutions are for the displacement-prescribed boundary condition (5).

Before proceeding with the traction boundary condition, we mention that Eq. (11) implies that the static displacement pulse should be "flat-topped" for a sinusoidal displacement boundary condition as noted by Narasimha et al. (6). For example, the static portion of the displacement given in (11) is $(\beta U k^2 / 8)x$. If the sample length is L, then the static portion of the displacement pulse in the time domain at the receiver end will be $(\beta U k^2 / 8)L$, which is obviously constant and is hence flat topped.

Next, we consider the case where, instead of displacement, a traction is prescribed on the boundary,

$$\sigma(0,t) = -\rho c \omega U \cos(\omega t).$$
(13)

The solution that satisfies the governing Eq. (4), the traction boundary condition (13), and the consistency condition (9) is given by

$$\frac{u_T}{U} = \sin\left[\omega\left(t-\frac{x}{c}\right)\right] - \frac{\beta U k^2}{8}\left(t - \frac{2x}{c}\right) - \frac{\beta U k}{16}\left[\sin\left[2\omega\left(t-\frac{x}{c}\right)\right]\right.$$
$$\left. - 2kx \cos\left[2\omega\left(t-\frac{x}{c}\right)\right]\right],$$
(14)

$$\frac{\sigma_T}{\rho c^2} = -Uk \cos\left[\omega\left(t-\frac{x}{c}\right)\right] + \frac{\beta U^2 k^3}{4} x \sin\left[2\omega\left(t-\frac{x}{c}\right)\right],$$
(15)

where the subscript T denotes that the solutions are for the traction-prescribed boundary condition (13)

As a longitudinal wave propagates through a medium, it causes cyclic compression and tension in the material. For most cases in linear elastic media, the magnitude of compressive and tensile strains is typically the same over each period. Thus the mean strain (or strain time-averaged over each period) vanishes. However, we note that the solution given in Eq. (14) leads to a non-zero mean strain,

$$\langle \varepsilon_T \rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \varepsilon_T dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[\frac{\partial u_r}{\partial x} + \frac{1}{2}\left(\frac{\partial u_r}{\partial x}\right)^2\right]dt$$
$$= \frac{U^2 k^2}{4}(1 + \beta).$$
(16)

In the literature, this is called the acoustic-radiation-induced strain. It is a consequence of the material and geometrical nonlinearities. We note also that, according to Eq. (15), the corresponding mean stress is zero, i.e.,

$$\langle \sigma_T \rangle \equiv \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sigma_T dt = 0.$$
(17)

Equations (16) and (17) seem to be at odds with each other in that a non-zero mean strain in an elastic body produces no mean stress. However, this can be explained if the non-zero mean strain $\langle \varepsilon_T \rangle$ is viewed as an eigenstrain.[14] Such an

eigenstrain is inelastic because it by itself produces no stress. In other words, when a time-harmonic displacement wave of amplitude U and frequency ω propagates through an elastic medium with quadratic nonlinearity, a static eigenstrain

$$\varepsilon^* = \frac{U^2 k^2}{4}(1+\beta) = \frac{U^2 \omega^2}{4c^2}(1+\beta) = -\frac{U^2 \omega^2}{4c^2}(2+\eta) \quad (18)$$

is generated in the solid. For many engineering materials, $\beta > 0$, so that $\varepsilon^* > 0$ leading to volumetric expansion.

This eigenstrain generated by an acoustic wave is analogous to the thermal strain generated by temperature change in a solid. In fact, the physical mechanisms of these two phenomena are indeed the same. They both are due to the lattice anharmonicity of the solid. This physical connection has been studied by Cantrell in the 1980s.[15–17] It was shown that the coefficient of thermal expansion is indeed related to the acoustic nonlinearity parameter β in a rather simple linear fashion. We note here that the thermodynamic driving force for thermal strain is temperature, and the thermodynamic driving force for acoustic-radiation-induced strain is the wave motion. It would be an interesting and useful endeavor to establish the relationship between these two different thermodynamic driving forces, thus the relationship between the thermal strain and the acoustic-radiation-induced strain.

Note that the total strain is the sum of the eigenstrain and the elastic strain ε^e (Ref. 14), i.e.,

$$\langle \varepsilon \rangle = \langle \varepsilon^e \rangle + \varepsilon^*. \quad (19)$$

It then follows from Eqs. (16) and (19) that when the traction is prescribed on the boundary, the elastic mean strain is zero, i.e., $\langle \varepsilon^e_T \rangle = 0$. This is consistent with Eq. (17) because only elastic mean strain causes mean stress.

Let us now consider the case when the displacement is prescribed on the boundary. By time-averaging Eq. (11), we have

$$\langle \varepsilon_D \rangle = \frac{U^2 k^2}{8}(2+\beta). \quad (20)$$

It then follows that the elastic mean strain in this case is

$$\langle \varepsilon^e_D \rangle = \langle \varepsilon_D \rangle - \varepsilon^* = -\frac{\beta U^2 k^2}{8}. \quad (21)$$

This elastic mean strain would generate mean stress according to Eq. (2) with the total strain being replaced by the elastic strain. To the leading order of $\beta U^2 k^2$, Eq. (2) gives

$$\langle \sigma_D \rangle = \rho c^2 \left[\langle \varepsilon^e_D \rangle + \frac{\beta}{2} \langle (\varepsilon^e_D)^2 \rangle \right] = \rho c^2 \langle \varepsilon^e_D \rangle$$
$$= -\rho c^2 \frac{\beta U^2 k^2}{8}. \quad (22)$$

This mean stress is called the acoustic-radiation-induced stress in the literature, which can also be obtained directly by carrying out the time average of Eq. (12).

The preceding analysis shows that the mean stress and the total mean strain are not intrinsic to the nonlinear behavior of the medium. Instead they are due to a combined effect of both the nonlinearity and boundary constraints. The material and geometrical nonlinearities only produce the eigenstrain, which by itself would not generate any stress. However, if the boundary is constrained so the medium cannot deform freely, an elastic mean strain must arise to accommodate the boundary condition. It is this elastic mean strain that generates the mean stress. When the boundary is not constrained, such as the case when the traction is prescribed on the boundary, the medium is free to shrink or expand in response to the eigenstrain ε^*, and no elastic mean strain is needed. Consequently, there will be no mean stress.

To further illustrate the preceding concept, let us consider another example, where the boundary condition is given by

$$u(0,t) = Vct + U\sin(\omega t), \quad (23)$$

where V is a given dimensionless constant on the order of $\beta U^2 k$.[2] The solution to Eq. (4) that satisfies the preceding boundary condition and the consistency condition (9) is given by

$$u_v = U\sin\left[\omega\left(t - \frac{x}{c}\right)\right] + Vct - \left(V - \frac{\beta U^2 k^2}{8}\right)x$$
$$+ \frac{\beta U k^2}{8} x \cos\left[2\omega\left(t - \frac{x}{c}\right)\right], \quad (24)$$

$$\frac{\sigma_v}{\rho c^2} = -Uk\cos\left[\omega\left(t - \frac{x}{c}\right)\right] - \left(\frac{\beta U^2 k^2}{8} + V\right)$$
$$- \frac{\beta U^2 k^2}{8}\left[\cos\left[2\omega\left(t - \frac{x}{c}\right)\right] - 2kx\cos\left[2\omega\left(t - \frac{x}{c}\right)\right]\right]. \quad (25)$$

The total mean strain is obtained by time-averaging Eq. (24)

$$\langle \varepsilon_V \rangle = \frac{U^2 k^2}{8}(2+\beta) - V. \quad (26)$$

The elastic mean strain follows from Eqs. (18) and (19),

$$\langle \varepsilon^e_V \rangle = \langle \varepsilon_V \rangle - \varepsilon^* = -\frac{\beta U^2 k^2}{8} - V. \quad (27)$$

Substituting Eq. (27) into Eq. (22) yields the mean stress

$$\langle \sigma_V \rangle = -\rho c^2 \left(\frac{\beta U^2 k^2}{8} + V\right). \quad (28)$$

Again, Eq. (28) can also be obtained by directly time-averaging Eq. (25)

Clearly, when $V = 0$, the preceding reduces to the case of displacement-prescribed boundary condition (5). When $V = -\beta U^2 k^2/8$, it is seen from Eqs. (26) to (28) that $\langle \varepsilon_V \rangle = \varepsilon^*, \langle \varepsilon^e_V \rangle = 0$, and $\langle \sigma_V \rangle = 0$, similar to the case of traction-prescribed boundary condition. In other words, even under displacement-prescribed boundary conditions, the

mean stress can still be zero if the boundary moves with a constant velocity of $V = -\beta U^2 k^2/8$.

It can be easily shown that for any V between $-\beta U^2 k^2/8 \leq V \leq 0$ (assuming $\beta > 0$), the mean strain and mean stress would vary between $(\beta U^2 k^2/4)(1 + \beta/2) \leq \langle \varepsilon_V \rangle \leq (\beta U^2 k^2/4)(1 + \beta) = \varepsilon^*$ and $-\beta U^2 k^2/8 \leq \langle \sigma_V \rangle / \rho c^2 \leq 0$. However, the eigenstrain $\varepsilon^* = (U^2 \omega^2/4c^2)(1 + \beta)$ remains the same for all these cases because the fundamental displacement waves have the same frequency ω and the same amplitude U.

One could also create a situation where $V = (U^2 k^2/8)(2 + \beta) = -(U^2 k^2/8)(1 + \eta)$, so that $\langle \varepsilon_V \rangle = 0$. Thus $\langle \varepsilon_V^e \rangle = -\varepsilon^*$ and $\langle \sigma_V \rangle = -\rho c^2 \varepsilon^*$. The positive V means that the boundary is being moved forward in the positive x direction at a constant velocity V in addition to the oscillatory motion. This particular static motion happens to balance the eigenstrain induced expansion, so there is no net mean strain.

In summary, as a fundamental wave of frequency ω and amplitude U passes through an elastic solid with quadratic nonlinearity, a tensile ($\beta > -1$) or a compressive ($\beta < 1$) eigenstrain $\varepsilon^* = (U^2 \omega^2/4c^2)(1 + \beta)$ is generated by the material and geometrical nonlinearities, causing a mean expansion/shrinkage of the region where the wave occupies. If the medium is not constrained (i.e., free to expand/shrink), the total mean strain is equal to the eigenstrain, and no mean stress (or radiation-induced stress) will be generated. This is the case when traction is prescribed on the boundary. When the medium is constrained, elastic mean strain will be generated, and the total mean strain is the sum of elastic mean strain and the eigenstrain. Because of the elastic mean strain, mean stress (or radiation-induced stress) is also generated in the medium.

Because only the eigenstrain is intrinsic to the deformation induced by the acoustic nonlinearity, and the total mean strain and the mean stress are functions of the boundary conditions, it is preferable that only the eigenstrain $\varepsilon^* = (U^2 \omega^2/4c^2)(1 + \beta)$ is called the acoustic-radiation-induced strain.

A number of authors, e.g., Refs. 1–6, have attempted to measure the acoustic nonlinearity parameter β by measuring the total mean strain. As shown here, the total mean strain depends not only on β but also on the boundary conditions. Because the ultrasonically measured mean strain is typically the total mean strain, to correctly interpret the data, boundary conditions for the ultrasonic test must be known; this is usually very difficult. For example, it is likely that a piezoelectric transducer attached to the sample imposes a boundary condition, that is, somewhere between the traction-prescribed and displacement-prescribed boundary conditions. However, the exact nature of the boundary constraints would depend on a number of factors including the acoustic impedance mismatch between the transducer and the material under measurement; this means that the precise boundary constraints will need to be calibrated for each material tested.

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