LETTERS TO THE EDITOR

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Pulse propagation in an elastic medium with quadratic nonlinearity (L)

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This letter examines the propagation of an acoustic pulse in an elastic medium with weak quadratic nonlinearity. Both a displacement pulse and a stress pulse of arbitrary shapes are used to generate the wave motion in the solid. By obtaining the explicit solutions for arbitrary pulse shapes, it is shown that for a sinusoidal tone-burst, in addition to a second order harmonic field, a radiation induced static strain field is also generated. These results help clarify some confusion in the recent literature regarding the shape of the propagating static displacement pulse.
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This letter examines the propagation of an acoustic pulse in an elastic medium with weak quadratic nonlinearity. To begin, consider a half-space defined by $x \geq 0$, where x is the Lagrangian (or material) coordinate describing the location of the material particle in the initial ($t = 0$) state. At any given time t, the displacement of the particle x from its initial position is denoted by $u(x,t)$. Deformation of the elastic body can then be described by the Lagrangian strain

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2. \tag{1}$$

We assume that the half-space is made of an elastic solid with quadratic nonlinearity, i.e., the normal (first Piola–Kirchhoff) stress is related to the Lagrangian strain/displacement gradient in the x-direction through

$$\sigma = \rho c^2\left[\varepsilon - \frac{\beta+1}{2}\varepsilon^2\right] = \rho c^2\left[\frac{\partial u}{\partial x} - \frac{\beta}{2}\left(\frac{\partial u}{\partial x}\right)^2\right], \tag{2}$$

where ρ is the mass density, c is the longitudinal phase velocity, and β is the acoustic nonlinearity parameter, all for the elastic solid in the undeformed (initial) state.

The displacement equation of motion governing the wave propagation in the x-direction is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

By a standard perturbation procedure, one may write the solution to Eq. (3) as

$$u(x,t) = u_1(x,t) + u_2(x,t), \tag{4}$$

where $|u_1(x,t)| \gg |u_2(x,t)|$, or $u_2 = O(u_1^2)$, and

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \; \frac{1}{c^2}\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = -\beta\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}. \tag{5}$$

The solution to the first expression of Eq. (5) that represents a forward propagating wave can be written as

$$u_1(x,t) = f(t - x/c). \tag{6}$$

It then follows that the second expression of Eq. (5) can be written as

$$\frac{1}{c^2}\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = g(t - x/c), \tag{7}$$

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where

$$g(s) = \frac{\beta}{c^3} f'(s) f''(s), \quad (8)$$

and the prime denotes the derivative with respect to the argument of the function. By a direct substitution, one can show that the solution to Eq. (7) is given by

$$u_2(x,t) = \frac{\beta x}{2c^2} \int_{0^+}^{t-x/c} f'(s) f''(s) ds + Dx + B(t - x/c), \quad (9)$$

where $B(y)$ is an arbitrary function of y and D is an integration constant, both need to be determined by the boundary conditions and/or the consistency condition[1,2]

$$\frac{\partial u}{\partial t} = \frac{2c}{3\beta} \left[\left(1 - \beta \frac{\partial u}{\partial x} \right)^{3/2} - 1 \right]. \quad (10)$$

If $f(s)$ is a smooth function for $s \in (0, t - x/c)$, the integral in Eq. (9) can be carried out,

$$u_2(x,t) = \frac{\beta x}{4c^2} \left([f'(t - x/c)]^2 - [f'(0^+)]^2 \right) + Dx + B(t - x/c). \quad (11)$$

This is the general solution to the second order governing Eq. (5).

Now, we determine $B(y)$ and D under different boundary conditions. First, consider the case where the displacement is prescribed on the boundary, i.e.,

$$u(0,t) = u_0(t). \quad (12)$$

Consequently,

$$u(0,t) = u_0(t), \; u_2(0,t) = 0. \quad (13)$$

It is then easy to show that

$$u_1(x,t) = f(t - x/c) = u_0(t - x/c). \quad (14)$$

The second order solution thus follows from Eq. (9) that

$$u_2 = \frac{\beta x}{4c^2} \left([f'(t - x/c)]^2 - [f'(0^+)]^2 \right), \quad (15)$$

where we have chosen $B(t) = 0$ in order to satisfy the boundary condition (13), and $D = \beta [f'(0)]^2 / (4c^2)$ to satisfy the consistency condition (10). Combining Eqs. (14) and (16) gives the solution under displacement boundary condition (12)

$$u_D(x,t) = u_0(t - x/c) + \frac{\beta x}{4c^2} \left([f'(t - x/c)]^2 - [f'(0^+)]^2 \right), \quad (16)$$

where the subscript D is to indicate that the solution is for the displacement prescribed boundary condition. Equation (16) was derived by Lamb[3] using a different method.

Next, consider the case where a traction is prescribed on the boundary,

$$\sigma(0,t) = \sigma_0(t). \quad (17)$$

Making use of the constitutive law (2), one may expand the stress into

$$\sigma(x,t) = \sigma_1(x,t) + \sigma_2(x,t), \quad (18)$$

where $|\sigma_1(x,t)| \gg |\sigma_2(x,t)|$ and

$$\sigma_1(x,t) = \rho c^2 \frac{\partial u_1}{\partial x}, \;\; \sigma_2(x,t) = \rho c^2 \left[\frac{\partial u_2}{\partial x} - \frac{\beta}{2} \left(\frac{\partial u_1}{\partial x} \right)^2 \right]. \quad (19)$$

The corresponding boundary conditions for $\sigma_1(x,t)$ and $\sigma_2(x,t)$ follow directly from Eqs. (17) and (18),

$$\sigma_1(0,t) = \sigma_0(t), \sigma_2(x,t) = 0. \quad (20)$$

Substituting Eq. (19) into Eq. (20) leads to

$$\left. \frac{\partial u_1(x,t)}{\partial x} \right|_{x=0} = \frac{\sigma_0(t)}{\rho c^2}, \;\; \left. \frac{\partial u_2}{\partial x} \right|_{x=0} = \frac{\beta}{2} \left(\frac{\partial u_1}{\partial x} \right)^2 \Bigg|_{x=0} = \frac{\beta}{2} \left(\frac{\sigma_0(t)}{\rho c^2} \right)^2. \quad (21)$$

In this case, it is straightforward to show that

$$u_1(x,t) = f(t - x/c) = \frac{1}{\rho c} \int_0^{t-x/c} \sigma_0(s) ds. \quad (22)$$

Substituting Eq. (22) into Eq. (11) in conjunction with the second expression of Eq. (21) leads to

$$D = \frac{\beta}{4\rho^2 c^4} [\sigma_0(0^+)]^2, \; B'(t) = \frac{-\beta [\sigma_0(t)]^2}{4\rho^2 c^3}. \quad (23)$$

Integrating the second expression of Eq. (23) yields

$$B(t) = -\frac{\beta}{4\rho^2 c^3} \int_{0^+}^{t} [\sigma_0(s)]^2 ds, \quad (24)$$

where we had ignored the integration constant, since a constant in the displacement is irrelevant for traction-prescribed problems.

Finally, combining Eqs. (22)–(24) and (11) gives the solution under the traction-prescribed boundary condition

$$u_T(x,t) = \frac{1}{\rho c} \int_0^{t-x/c} \sigma_0(s) ds + \frac{\beta x}{4\rho^2 c^4} [\sigma_0(t - x/c)]^2 - \frac{\beta}{4\rho^2 c^3} \int_{0^+}^{t-x/c} [\sigma_0(s)]^2 ds. \quad (25)$$

To elucidate some physical features of the solution obtained above, consider the propagation of a sinusoidal pulse of angular frequency ω. For convenience, define a rectangular pulse

$$P(t) = \mathrm{H}(t)\mathrm{H}(\tau - t), \tag{26}$$

where $\mathrm{H}(t)$ is the Heaviside step function, and $\tau = 2n\pi/\omega$ with n being a positive integer.

Now, let us begin with the displacement-prescribed boundary condition

$$u_0(t) = UP(t)\sin\omega t. \tag{27}$$

Substituting Eq. (27) into Eq. (16) gives

$$u_D(x,t) = \left[U\sin\omega\left(t - \frac{x}{c}\right) + \frac{\beta U^2\omega^2 x}{8c^2}\cos 2\omega\left(t - \frac{x}{c}\right) + \frac{\beta U^2\omega^2 x}{8c^2}\right]P\left(t - \frac{x}{c}\right). \tag{28}$$

We note that the first term on the right hand side of Eq. (28) is the original propagating pulse of frequency ω. The second term represents a propagating pulse of frequency 2ω with linearly growing amplitude. The third term, $(\beta U^2\omega^2 x/8c^2)$ $P(t - x/c)$ is the static portion of the displacement. It represents a propagating static pulse in that (1) at any fixed location x, the displacement is a rectangular pulse in the time domain, thus the term pulse, (2) at any fixed time t, the medium between $x = c(t-\tau)$ and $x = ct$ under goes a positive uniform strain, i.e., the displacement increases linearly from $x = c(t-\tau)$ to $x = ct$, thus the term "static," and (3) this region of uniform strain moves in the positive x-direction with velocity c, thus the term propagating.

Further, we note that the amplitude of the static displacement at a given point is proportional to the distance between this point and the boundary, and the proportional constant is $\beta U^2\omega^2/8c^2$. If the signal for the static portion of the displacement is recorded by a receiver at location x_0, the recorded signal plotted as a function of time will be a "flat topped" rectangle of height $(\beta U^2\omega^2/8c^2)x_0$. The length of the rectangle will be τ. This is consistent with the experimental observations of Refs. 4 and 5 and the numerical analysis based on the finite difference method.[6] We note that the numerical analysis in Ref. 6 is indeed for displacement-prescribed boundary condition.

Also, for the time-harmonic case where $\tau \to \infty$, one may reduce Eq. (28) to the time-harmonic solution obtained in Ref. 1,

$$u_D(x,t) = \left[U\sin\omega\left(t - \frac{x}{c}\right) + \frac{\beta U^2\omega^2 x}{8c^2}\cos 2\omega\left(t - \frac{x}{c}\right) + \frac{\beta U^2\omega^2 x}{8c^2}\right]H\left(t - \frac{x}{c}\right). \tag{29}$$

Next, consider the traction-prescribed boundary condition

$$\sigma_0(t) = -\rho c\omega UP(t)\cos\omega t. \tag{30}$$

Substituting Eq. (30) into (25) yields

$$u_T(x,t) = U\sin\left[\omega\left(t - \frac{x}{c}\right)\right]P(t - x/c) - \frac{\beta U^2\omega^2 x}{8c^2}(2x - ct)P(t - x/c) + \frac{\beta U^2\omega}{16c}\left[\frac{2\omega x}{c}\cos 2\omega(t - x/c) - \sin 2\omega(t - x/c)\right]P(t - x/c). \tag{31}$$

Clearly, the second term on the right hand side of Eq. (31) represents the static displacement. As in the case of displacement-prescribed boundary condition, the static displacement is also a propagating pulse with its amplitude growing with propagation distance. However, if the signal for the static portion of the displacement is recorded by a receiver at location x_0, the recorded signal plotted as a function of time will not be "flat" topped. Instead, the top of the pulse will linearly decrease from $(\beta U^2\omega^2/8c^2)x_0$ at the front edge to $(\beta U^2\omega^2/8c^2)(x_0 - \tau c)$ at the trailing edge of the pulse. In other words, if the signal for the static portion of the displacement is recorded by a receiver at location x_0, the recorded signal plotted as a function of time will not be a "flat topped" rectangle. Instead, it will be a "slant topped" trapezoid. Our results are in sharp contradiction with the original predictions of Yost and Cantrell[7,8] who correctly predicted that the static strain is a flat topped pulse of magnitude $\beta U^2\omega^2/(8c^2)$, but then incorrectly suggested that the static displacement is a right-angle triangle with a peak value of $\beta U^2\omega^2/(8c^2)L$, where $L = c\tau$ is the spatial length of the pulse. In fact, causality would dictate that the pulse shape cannot be a right-angle triangle predicted in Refs. 7 and 8. Information about the length of the pulse generated at $x = 0$, $t = \tau$ cannot propagate faster than the velocity of sound, therefore it cannot influence the initial peak value of the quasi-static pulse. Further, the amplitude of the static pulse must increase with propagation distance for a given pulse length since the nonlinear propagating part of the effect is cumulative.

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