A microcrack-based continuous damage model for brittle geomaterials

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Abstract

A new microcrack-based continuous damage model is developed to describe the behavior of brittle geomaterials under compression dominated stress fields. The induced damage is represented by a second rank tensor, which reflects density and orientation of microcracks. The damage evolution law is related to the propagation condition of microcracks. Based on micromechanical analyses of sliding wing cracks, the actual microcrack distributions are replaced by an equivalent set of cracks subjected to a macroscopic local tensile stress. The principles of the linear fracture mechanics are used to develop a suitable macroscopic propagation criterion. The onset of microcrack coalescence leading to localization phenomenon and softening behavior is included by using a critical crack length. The constitutive equations are developed by considering that microcrack growth induces an added material flexibility. The effective elastic compliance of damaged material is obtained from the definition of a particular Gibbs free energy function. Irreversible damage-related strains due to residual opening of microcracks after unloading are also taken into account. The resulting constitutive equations can be arranged to reveal the physical meaning of each model parameter and to determine its value from standard laboratory tests. An explicit expression for the macroscopic effective constitutive tensor (compliance or stiffness) makes it possible, in principal, to determine the critical damage intensity at which the localization condition is satisfied. The proposed model is applied to two typical brittle rocks (a French granite and Tennessee marble). Comparison between test data and numerical simulations show that the proposed model is able to describe main features of mechanical behaviors observed in brittle geomaterials under compressive stresses. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Most brittle geomaterials (rocks and concrete) are subject to induced damage by oriented microcracks under compressive stresses. The onset and growth of microcracks significantly affect the macroscopic behavior of material and the main consequences can be summarized as follows:

- non-linearity of stress–strain relations;
- deterioration of elastic properties;
- induced material anisotropy;
- significant volume dilatation;
- irreversible damage strains due to residual crack opening;
- unilateral response due to crack closure effect;
- plastic strains related with sliding phenomenon and plastic crack tip zone;
hysteresis loop associated with frictional mechanism.

These features have to be considered in constitutive models of brittle geomaterials. The first problem to be solved is the evaluation of effective properties of material containing microcracks. One natural approach is from micromechanical considerations. Contributions of individual microcracks to material macroscopic flexibility and the effects due to interaction between microcracks are calculated, then averaged to determine the effective elastic constants of cracked elastic solids. The most widely used averaging schemes are the self-consistent method and the differential scheme (Budiansky and O'Connell, 1976; Hashin, 1988). An alternative approach is to deduce the effective elastic constants from general expressions of the Gibbs free energy of cracked solids as a function of stress and crack density tensors (Sayers and Kachanov, 1991; Kachanov, 1992, 1993). Based on these methods, micromechanical constitutive models have been developed to describe overall response and damage evolution of brittle materials (examples include Fanella and Krajcinovic, 1988; Nemat-Nasser and Obata, 1988; Ju and Lee, 1990; Lee and Ju, 1990; Ju and Tseng, 1992, Ju and Chen, 1994; Prat and Bazant, 1996). The key point in these models is to determine relevant nucleation and propagation criteria of microcracks and kinetic laws in microscopic level (in REV). The main advantage of such models is the ability to account for physical mechanisms involved in nucleation and growth of microcracks. However, the macroscopic behavior of material is obtained through a homogenization procedure. This renders the models difficult to be used in practical applications.

On the other hand, phenomenological damage models have been developed using internal variables to represent material damage state (Lemaitre, 1992; Chow and Wang, 1987a,b; Ju, 1989; Halm and Dragon, 1996, 1998; Hayakawa and Kurakami, 1997; Murakami and Kamiya, 1997; Swoboda and Yang, 1999a,b, only to mention some more recent models). The advantages of such models are that they are formulated in the irreversible thermodynamics framework and provide macroscopic constitutive equations, which can be easily applied to engineering analyses. The weakness is that some concepts and parameters used are not clearly connected to physical mechanisms. For example, one class of phenomenological models is based on the concept of effective stress initially proposed by Kachanov for metal creep failure (Lemaitre, 1992). Although the physical meaning of this concept is clear in isotropic case, the direct extension to the anisotropic case leads to a non-symmetric stress tensor. Different mathematical techniques have been proposed to obtain a symmetric effective stress tensor whose physical meaning is questionable. In addition, many phenomenological models use a tensile strain-based damage criterion. Laboratory results on brittle rocks reveal that such a criterion may overestimates the initial damage threshold in triaxial compression condition with high confining pressures. Phenomenological models using stress-based damage criterion are also available. However the performance of these models to predict damage of brittle geomaterials under compression-dominated stress fields is not fully proved. A variant approach to develop continuous damage models has been proposed by Ortiz (1985) and extended by Yazdani and Schrecher (1988, 1989). Their theory is based on the determination of a kinetic theory for damage induced strains. However, the material damage state is not properly defined by a physically based state variable. One alternative way is to develop continuous damage models based on the relevant results of microscopic analyses. Costin (1985) has proposed a microcrack-based model by proposing an equivalent macroscopic damage criterion to represent microcrack propagation. A modified version of this model has been applied to different rocks by Rudnicki and Chau (1996). Although this model is able to predict basic mechanical behaviors of brittle rocks in laboratory tests level, the application to practical engineering problems is not easy. The principal reason is that the effective elastic tensor of damaged material is not explicitly formulated. In this paper, we propose a new microcrack-based continuous model to overcome the difficulties encountered in the Costin's model.
The present model intends to combine principal advantages of phenomenological and micromechanical models. The induced damage is represented by a second rank tensor, which reflects density and orientation of microcracks. The damage evolution law is related to the propagation condition of microcracks. The consequences of actual microcrack distributions on macroscopic behaviors are considered to be equivalent to those of a fictive set of cracks subjected to a macroscopic local tensile stress. The effective elastic compliance tensor is derived from a suitable Gibbs free energy function. Further, irreversible damage strains due to residual opening of microcracks after unloading are also taken into account. A particular effort is made so that each model parameter corresponds to a physical meaning and can be identified from measurable data in laboratory tests. However, unilateral effects related to microcrack closure are not considered in the present model (Chaboche, 1992). Finally, the proposed model is applied to two typical brittle rocks (a French granite and Tennessee marble). Comparison between test data and numerical simulations show that the proposed model is able to describe main features of mechanical behaviors observed in most geomaterials under compression dominated stresses.

2. Presentation of the microcrack-based continuous damage model

2.1. Basic consideration

Nucleation and growth of microcracks lead to deterioration of macroscopic elastic properties. Two principal consequences are the added elastic flexibility (or diminution of elastic stiffness) and induced material anisotropy. Further, damage-related irreversible deformations can develop due to residual opening of microcracks after unloading. In addition, pure plastic flow can also develop by dislocation motions and sliding phenomena along some preferential planes. However, in most brittle rocks, the plastic flow has a smaller importance with respect to damage development. Therefore, it is neglected in the present model although the extension of the model to include the plastic flow does not have any basic difficulties. Accordingly, the constitutive equations of the damaged material can be written in the following general form:

\[ \varepsilon - \varepsilon'(D) = S(D) : \sigma = (S^0 + S^c(D)) : \sigma, \]  

where \( S(D) \) is the fourth-order symmetric effective elastic compliance tensor of damaged material which is the sum of the initial compliance tensor \( S^0 \) in the undamaged state and the added compliance tensor \( S^c(D) \) due to microcracks. The internal variable \( D \) is used to characterize damage state of material. The second symmetric tensor \( \varepsilon'(D) \) denotes the inelastic damage strain which remains upon reduction of the stress to zero. Therefore, the principal elements to be completed are the determination of evolution law for damage variable, the evaluation of the added compliance tensor and inelastic damage strain.

2.2. Microcrack-based damage evolution law

The main purpose of the present work is the modeling of anisotropic damage in brittle geomaterials which are subject to compression dominated stresses. In order to account for extent and orientation of microcracks, a second rank symmetric tensor is used. From the micromechanical point of view, the damage tensor can be determined from the extent and orientation of each microcrack involved in the REV (Kachanov, 1992, 1993; Lubarda and Krajcinovic, 1993)

\[ D = \sum_k d^k(A) (n \otimes n)_k, \]  

where \( d^k(A) \) represents the crack density corresponding to the decohesion area \( A \) and \( n \) is the unit normal of the crack. If we suppose that the microdefects can be idealized as penny-shaped microcracks, then the crack density tensor can be expressed by

\[ D = \frac{1}{V} \sum_k (l^k n \otimes n)_k, \]
where \( V \) is the volume of the REV and \( l \) the radius of microcracks. Most geomaterials (rocks and concrete) contain initial microdefects due to their formation history and fabrication process. However, a precise characterization of the initial microcrack distribution is usually not easy. In this study, a randomly oriented set of initial microdefects is assumed. Therefore, the induced damage tensor is characterized by the relative variation of microcrack density with respect to the initial state:

\[
D = \sum_k \left( \frac{l^3 - l_0^3}{b^3} n \otimes n \right)_k
= \sum_k \left( (r^3 - r_0^3)n \otimes n \right)_k, \tag{4}
\]

where \( l_0 \) is the average radius of initial microcracks and \( b \) denotes a characteristic length at the onset of accelerated microcrack interaction.

With this definition of damage tensor, the damage evolution law should be determined from propagation conditions and kinetic laws of microcracks. Many experimental and theoretical studies have been performed to understand principal propagation modes of microcracks in brittle materials under compressive stresses (for example, Wawersik and Brace, 1971; Nemat-Nasser and Horii, 1982; Wong 1982; Horii and Nemat-Nasser, 1983, 1985, 1986, 1993; Steif, 1984; Sammis and Ashby, 1986; Fredrich and Wong, 1986; Fredrich et al., 1989). The most widely used propagation mode is the idealized sliding wing crack model. This model includes the sliding mechanism along the existing inclined crack and the tensile mechanism on the opened wing cracks. The propagation condition is then controlled by the normal and shear stresses applied on the crack at the microscale.

Since a continuous damage model is concerned in this work, a realistic equivalent macroscopic propagation criterion should be found. For this purpose, it is assumed that the influences of actual microcracks on macroscopic behavior can be equivalently represented by those of a fictive planar crack which is subjected to a macroscopic local tensile stress and propagate in tensile mode. The expression of this tensile stress is, in general, complex. However, most triaxial compression tests on brittle rocks suggest that the majority of microcracks propagate in the direction normal to the minimum compressive stress. The propagation condition is controlled by deviatoric stress and influenced by confining pressure. Therefore, the following macroscopic propagation criterion consistent with linear fracture mechanics theory is proposed:

\[
F(\sigma, r) = \sqrt{r} \left[ \sigma_n + f(r) n \cdot \sigma^d \cdot n \right] - c_r = 0, \tag{5}
\]

where \( \sigma_n = n \cdot \sigma \cdot n \) is the normal stress applied to the crack with the unit normal \( n \). The second-order tensor \( \sigma^d = \sigma - (\sigma_{dd}/3) I \) denotes the deviatoric stresses. The parameter \( c_r \) represents the material toughness to microcrack propagation. The scalar valued function \( f(r) \) defines the proportion of the applied field stress that is transmitted to the local concentration of tensile stress. The expression of this function may be determined approximately from relevant numerical results of micromechanical models. The general form of the function must, however, satisfy certain requirements. For small crack extents, it should decrease, reflecting the relaxation of local tensile stress as the crack grows away from the source; as the crack length becomes large enough (when \( r = 1 \)) to interact with the stress fields of other nearby cracks \( f(r) \) increases. The first effect causes initially stable growth and the second marks the onset of accelerated crack interaction producing a peak stress in stress–strain curves. The following particular form has these basic features:

\[
f(r) = t \left[ 1 - \frac{(1 - r)^2}{r_0(r_0 - 1)} \right], \tag{6}
\]

where \( t \) is a parameter of the model. The damage surface associated with the set of microcracks in the orientation \( n \) is given by

\[
(n \cdot \sigma^d \cdot n) = \frac{c_r}{f(r)\sqrt{r}} - \frac{1}{f(r)} \sigma_n. \tag{7}
\]
The criterion (5) is modified from the initial form proposed by Costin (1985) and has the same form as the criterion used by Rudnicki et al. (1996). However, the physical interpretation is clearly different. In the present model, the criterion (5) plays as the macroscopic damage criterion for a given loading orientation and the dimensionless crack length \( r = l/b \) should be considered as the macroscopic variable representing the local damage intensity in this orientation.

At this stage, the current components of the damage tensor can be determined from Eqs. (4)–(6) by summation of contributions of each crack. But because this kind of calculation is usually long to perform, some simplifications are made. According to the numerical results reported by Jeyakumaran and Rudnicki (1995) who studied the propagation conditions of microcracks in different orientations in triaxial compression tests, crack growth first occurs in the axial direction, and as the deviatoric stress increases, crack growth occurs in a cone of directions centered on the axis and including an angle that increases to about 10–15° at peak stress. Most laboratory observations confirm this kind of microcrack growth in brittle rocks under compressive stresses (Nemat-Nasser and Horii, 1982; Wong, 1982; Sammis and Ashby, 1986; Fredrich and Wong, 1986, for instance). These results mean that material damage is dominated by the set of cracks oriented nearly in the direction of the maximum compression. Consequently, we assume that the material damage is only contributed by three equivalent orthogonal sets of cracks which are parallel to the principal directions of the deviatoric stress tensor \( V^k \ (k = 1, 2, 3) \).

\[
D = \sum_{k=1}^{3} ((r^3 - r_0^3)V \otimes V)_k. \tag{8}
\]

The consequence of this simplification in triaxial compression tests is that only the set of microcracks in the axial direction is considered in the calculation of the damage tensor. However, the contributions of the other microcracks to macroscopic behaviors (for instance, the diminution of the axial elastic modulus) will not be neglected. It will be shown that their contributions will be indirectly taken into account in the determination of the effective elastic compliance tensor.

It is useful to notice that the proposed model predicts an unstable microcrack propagation in uniaxial (or multiaxial) tension and leads to a perfect elastic–brittle behavior of material. This is in agreement with most experimental data for brittle rocks. The uniaxial tensile strength is given by the following relation:

\[
\sigma_t = \frac{c_t}{\sqrt{F_0(1 + (2r/3r_0))}}. \tag{9}
\]

2.3. Determination of elastic compliance tensor

Since we are developing a continuous damage model here, an energy-based approach is used to determine the effective elastic compliance tensor. Again, some simplifying assumptions are adopted. The undamaged material is assumed to have a linear elastic behavior. The material response is also assumed to be linear upon unloading at a constant state of damage. In addition, the influence of interaction between microcracks on the free energy function is assumed to be small so that the latter can be expressed as a linear function of the damage tensor. Based on the previous works of Hayakawa and Murakami (1997), the following particular form of the Gibbs free energy function is used

\[
G(\sigma, D) = \frac{1 + v_0}{2E_0} \text{tr}(\sigma \cdot \sigma) - \frac{v_0}{2E_0} (\text{tr}\sigma)^2 + a_2 \text{tr}(\sigma \cdot D) + a_3 \text{tr}(\sigma) \cdot \text{tr}(\sigma_D) + a_4 \text{tr}(D) \cdot \text{tr}(\sigma \cdot \sigma), \tag{10}
\]

where three parameters (\( a_2, a_3, \) and \( a_4 \)) are introduced to characterize microcrack contributions to the free energy of the material. \( E_0 \) and \( v_0 \) are the initial elastic modulus and the Poisson’s ratio of undamaged material. Differentiation of the free energy function leads to the effective strain–stress relations.
\[ \varepsilon - \varepsilon^e(D) = \frac{\partial G(\sigma, D)}{\partial \sigma} \]

\[ = 1 + \frac{v_0}{E_0} \sigma - \frac{v_0}{E_0} (\text{tr} \sigma) I + a_2(\sigma \cdot D + D \cdot \sigma) + a_3(\text{tr} \sigma D) I + (\text{tr} \sigma) D \]

\[ + 2a_4(\text{tr} D) \sigma. \quad (11) \]

The constitutive equation (11) can be rewritten in the following form:

\[ e_{ij} - e^e_{ij}(D) = S_{ijkl}(D)\sigma_{kl}, \quad (12) \]

where the effective elastic compliance tensor of damaged material is given explicitly by:

\[ S_{ijkl}(D) = \frac{1 + \frac{v_0}{E_0} \sigma}{2E_0} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) - \frac{v_0}{E_0} \delta_{ij} \delta_{kl} \]

\[ + \frac{1}{2} a_2 (\delta_{ik} D_{jl} + \delta_{il} D_{jk} + D_{ik} \delta_{jl} + D_{il} \delta_{jk}) \]

\[ + a_3 \left( \text{tr} D \delta_{ij} \delta_{kl} \right) \]

\[ + a_4 \text{tr} D (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (13) \]

In the co-ordinates system associated with the principal directions of the damage tensor, the strain–stress equations can be expressed in the standard matrix form by using the Voigt notation

\[
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{12} \\
2\varepsilon_{23} \\
2\varepsilon_{31}
\end{pmatrix}
= \begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{pmatrix}
\]

\[ = \begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{44} \\
\sigma_{45} \\
\sigma_{55}
\end{pmatrix}. \quad (14) \]

The components of the effective elastic compliance tensor are given as functions of principal values of damage tensor as follows:

\[ S_{11} = \frac{1}{E_0} + (2a_2 + 2a_3)D_1 + 2a_4 \text{tr} D, \]

\[ S_{22} = \frac{1}{E_0} + (2a_2 + 2a_3)D_2 + 2a_4 \text{tr} D, \]

\[ S_{33} = \frac{1}{E_0} + (2a_2 + 2a_3)D_3 + 2a_4 \text{tr} D, \]

\[ S_{12} = S_{21} = -\frac{v_0}{E_0} + a_3(D_1 + D_2), \]

\[ S_{13} = S_{31} = -\frac{v_0}{E_0} + a_3(D_1 + D_3), \]

\[ S_{23} = S_{32} = -\frac{v_0}{E_0} + a_3(D_2 + D_3), \]

\[ \frac{1}{2G_{12}} = \frac{1 + \frac{v_0}{E_0} + a_2(D_1 + D_2) + 2a_4 \text{tr} D}{G_{12}}, \]

\[ \frac{1}{2G_{23}} = \frac{1 + \frac{v_0}{E_0} + a_2(D_2 + D_3) + 2a_4 \text{tr} D}{G_{23}}, \]

\[ \frac{1}{2G_{31}} = \frac{1 + \frac{v_0}{E_0} + a_2(D_3 + D_1) + 2a_4 \text{tr} D}{G_{31}}. \]

Laboratory investigations on many brittle rocks subjected to compressive stresses indicate that principal elastic moduli decrease and the Poisson’s ratios increase as microcracks grow. These observations place the following physical bounds on the values of three model parameters:

\[ a_2 \geq 0, \quad a_4 \geq 0, \quad a_3 \leq 0, \quad a_2 + a_3 > 0. \quad (16) \]

Actually, the last condition is not absolutely necessary but is satisfied in practice. Let us imagine a solid containing one set of co-linear microcracks. Obviously the effective elastic modulus in the direction normal to the plane of microcracks should be significantly smaller than that in the direction parallel to microcracks. Such a result follows from the last condition.

We can notice that the term with \( \text{tr}(D) \) in the free energy function represents certain isotropic effects of induced damage. The effective elastic modulus in one principal direction not only depends on the principal damage value in its own direction but also on the values in the other two orthogonal directions. This feature is essential in the present microcrack-based model with respect to the simplifications made in the damage evolution law. Indeed, the damage evolution is assumed to depend on the propagation of cracks oriented in the tensile principal directions of deviatoric stress.
tensor. The contributions of the others cracks to macroscopic behavior are indirectly taken into account thanks to this isotropic effect. For example, in triaxial compression, the damage component in the axial direction is zero as only microcracks parallel to the axis are considered in the damage evolution law. However, the axial elastic modulus does not remain constant and slightly decreases as cracks grow. This is consistent with experimental observations.

2.4. Determination of inelastic damage strain

The inelastic damage strains \( \varepsilon^r(D) \) correspond to residual opening of wing cracks and misfit of crack surfaces after unloading of applied stresses. Therefore, it is directly related to the coupled frictional sliding and extension mechanism. The frictional effect is much more important under compressive stresses than in tensile stresses. Accordingly, it is reasonable to assume that the inelastic damage strains develop under compressive stress only, as suggested by Yasdani and Schreyer (1988). For this purpose, the total stress tensor is decomposed into a positive cone and a negative cone, as proposed by Ortiz (1985):

\[
\sigma = \sigma^+ + \sigma^-,
\]

\[
\sigma^+ = P^+(\sigma) = \sum_k H(\sigma_k)\sigma_k V^k \otimes V^k,
\]

\[
\sigma^- = P^-(\sigma) = \sum_k H(-\sigma_k)\sigma_k V^k \otimes V^k,
\]

where \( P^+ \) and \( P^- \) are two operators which, respectively, remove the negative and positive eigenvalues of a second-order tensor, and \( H(x) \) is the Heaviside step function. Furthermore, as the propagation of a sliding wing crack is caused by the deviatoric compressive stresses, it is assumed that the inelastic damage strain is associated with the deviatoric part of the compressive stress tensor noted here \( \sigma^{-d} \).

The rate of inelastic damage strain is finally expressed as follows on the basis of the previous works of Yazdani and Schreyer (1988):

\[
\frac{d\varepsilon^r(D)}{dt} = \frac{\omega}{E_0} \text{tr}(dD)(\beta S^+ + S^-),
\]

where the parameter \( \omega \) defines the proportionality between inelastic damage strain rate and the damage rate. Two second-order tensor \( S^+ \) and \( S^- \) are, respectively, the positive and negative cones of the deviatoric compressive stress tensor \( \sigma^{-d} \):

\[
S^+ + S^- = \sigma^{-d}, \quad S^+ = P^+(\sigma^{-d}),
\]

\[
S^- = P^-(\sigma^{-d}).
\]

The above decomposition is essential and allows to describe irreversible volume dilatation due to residual crack opening when the parameter \( \beta > 1 \). When \( \beta = 1 \), the inelastic damage strains develop without irreversible volume change and \( \beta < 1 \) leading to compressive inelastic volumetric strain which usually does not appear in brittle rocks under compressive stresses.

The last point to be discussed concerns the theoretical weakness of the model. It is not fully formulated in the framework of irreversible thermodynamics in contrast to classical phenomenological models. The second principle is not automatically verified and should be checked step by step during numerical computation. It is expressed by the positiveness of damage dissipation:

\[
Y^d : D \geq 0,
\]

where the conjugate force associated with the damage tensor is derived from the Gibbs free energy potential:

\[
Y^d = \frac{\partial G(\sigma, D)}{\partial D} = a_2(\sigma \cdot \sigma) + a_3(tr\sigma)\sigma + a_4 tr(\sigma \cdot \sigma)I.
\]

3. Application

In this section, the procedure for the determination of model parameters is first presented. Then the proposed model is applied to describe the behavior of two typical brittle rocks: a French granite and the Tennessee marble. The French granite was investigated in the framework of the jointed research project "GDR-FORPRO" supported by the CNRS and ANDRA. The purpose
was to study influences of induced damage in safety analysis of underground storage of radioactive wastes. Laboratory investigations have shown important microcrack induced anisotropic damage in this material. On the other hand, inelastic behaviors related to microcrack growth and induced anisotropy in Tennessee marble have been investigated by Olsson (1995), Rudnicki and Chau (1996), Rudnicki et al. (1996).

3.1. Determination of model’s parameters

The most usual tests performed in rock mechanics are different kinds of triaxial tests on cylinder samples and with axisymmetric loading condition. For the co-ordinates system shown in Fig. 1, the following conditions are satisfied:

\[ \sigma_2 = \sigma_3, \quad \epsilon_2 = \epsilon_3, \quad (22a) \]

\[ D_2 = D_3 = r^3 - r_0^3, \quad D_1 = 0. \quad (22b) \]

**Determination of parameters for damage surface** \((t, r_0, c_r)\): These parameters are determined from peak stress and damage initiation stress in triaxial compression tests with different confining pressures. In this case, the initial damage surface (point A in Fig. 1) and failure (peak stress in Fig. 1) surface are expressed by:

\[ (\sigma_3 - \sigma_1)_{\text{init}} = \frac{3r_0c_r}{t \sqrt{r_0}} + \frac{3r_0}{t} (-\sigma_3), \quad (23) \]

\[ (\sigma_3 - \sigma_1)_{\text{peak}} = \frac{3c_r}{f(r_p) \sqrt{r_p}} - 3 \frac{f(r_p)}{f(r_p)} \sigma_3, \quad (24a) \]

\[ (\sigma_3 - \sigma_1)_{\text{peak}} \approx \frac{3c_r}{t} + 3 \frac{3}{t} (-\sigma_3). \quad (24b) \]

The crack length \(r_p\) corresponding to the peak stress is theoretically determined by \(\partial F(\sigma, r) / \partial r = 0\) from Eq. (5). However, this condition nearly coincides with that for accelerated crack interaction, i.e., \(\partial f(r) / \partial r = 0\) for \(r = 1\) from Eq. (6). Therefore, one can take \(r_p \approx 1\) as a reasonable approximation of the failure surface, as described by Eq. (24b). Consequently, the three parameters are determined from the slopes and intercepts of the experimental damage initiation and failure lines in the triaxial plane. We have four equations for three independent parameters to be determined. However, the damage initiation stresses are usually difficult to capture because the transition from linear part to non-linear part is not obvious in actual stress–strain curves. Therefore, it is suggested to first determine the parameters \((t, c_r)\) from the failure surface and then find the value of \(r_0\) to obtain the best approximation of the damage initiation surface.

For Tennessee marble, experimental results (see Figs. 2–4) show a smooth transition from hardening regime to strain softening. This is possibly related to certain ductile plastic mechanisms. In order to approximately capture such a smooth transition, we have used a slight modified form for the local tensile stress proportion function \(f(r)\) (see Eq. (6)), which is expressed by

\[ f(r) = b_1 - b_2r \exp(1 - r), \quad (25) \]

where \(b_1\) and \(b_2\) are two model parameters. The general procedure of determination presented above is not changed. We have now four parameters to be determined from the slopes and intercepts of the failure and damage initiation lines.
Determination of parameters in free energy function \((E_0, m_0, a_2, a_3, a_4)\): The initial elastic constants \((E_0, m_0)\) are determined from the linear part of the idealized stress–strain curve in Fig. 1. In many rocks, there is a non-linear part at the beginning of loading due to closure of initial defects, followed by a linear phase which presents the actual elastic response of the material. Three parameters \((a_2, a_3, a_4)\) are directly related to the variation of elastic properties in different principal directions. In a conventional triaxial compression test, only the axial modulus \(E_1\) and dilatation coefficient \(v_{13}\) can be measured. The following expressions give the values of \((a_3, a_4)\):

\[
a_3 = \frac{1}{D_3} \left( -\frac{v_{31}}{E_1} + \frac{v_0}{E_0} \right),
\]

\[
a_4 = \frac{1}{4D_3} \left( \frac{1}{E_1} - \frac{1}{E_0} \right).
\]

In these relations, the current elastic constants are determined by unloading while the damage value is obtained from the consistency condition of the damage criterion. The parameter \(a_2\) is related to the variation of the elastic modulus in the (radial) direction normal to the plane of microcracks. It can be determined from experimental measurement of this modulus from a true triaxial test if available. In the absence of such data yet, it can be determined by fitting experimental response in a radial compression or extension test. Another alternative is to use numerical results from micromechanical models to compare the ratio between the effective moduli in the directions normal and parallel to the plane of microcracks.

**Determination of parameters for inelastic damage strain \((\omega, \beta)\):** When the parameters involved in damage criterion and free energy function are determined, the fully brittle elastic damage responses can be predicted using the constitutive equations. If such responses agree with experimental data, it means that the inelastic damage strain \(\varepsilon(D)\) can be neglected \((\omega = 0, \beta = 0)\). Else, their values are easily fitted by reducing the difference between the theoretical elastic damage response and experimental data.

### 3.2. Numerical simulations

In this section, the proposed model is applied to simulate the responses of the granite and the Tennessee marble. The values of parameters obtained for two materials are listed in Tables 1 and 2.

**Table 1**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_0) (MPa)</td>
<td>82300</td>
</tr>
<tr>
<td>(a_1) (MPa(^{-1}))</td>
<td>(-4.55 \times 10^{-6})</td>
</tr>
<tr>
<td>(b_1)</td>
<td>8.35</td>
</tr>
<tr>
<td>(\omega)</td>
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<tr>
<td>(m_0)</td>
<td>0.3</td>
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<tr>
<td>(a_2) (MPa(^{-1}))</td>
<td>(1.02 \times 10^{-4})</td>
</tr>
<tr>
<td>(a_3) (MPa(^{-1}))</td>
<td>(2.0 \times 10^{-5})</td>
</tr>
<tr>
<td>(b_2)</td>
<td>7.1</td>
</tr>
<tr>
<td>(\beta)</td>
<td>2.5</td>
</tr>
<tr>
<td>(r_0)</td>
<td>58</td>
</tr>
<tr>
<td>(\gamma_0)</td>
<td>0.35</td>
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</table>
For the Tennessee marble, only conventional triaxial compression tests are considered. Three tests with unloading–reloading cycles are available and the confining pressures are, respectively, 3.44, 8.29 and 20.6 MPa.

Numerical simulations are presented in Figs. 2–4 and compared with experimental data. There is a good agreement for both overall stress–strain curves and variations of effective elastic properties. The volumetric dilatancy is also correctly predicted. However hysteresis unloading loops are not predicted in the present model and the unloading responses are simplified by a straight line whose slope represents the current elastic modulus. Furthermore, some discrepancy is still observed between predicted strains and experimental ones in the peak zone although a modified form of the function \( f(r) \) was used (see Eq. (23)). Indeed, the numerical prediction provides a typical response of a brittle rock marked by a peak point and softening behavior after peak stress. In contrast, experimental data suggest a much more ductile behavior similar to perfect plastic responses. However, the smooth transition from hardening to softening is quite correctly reproduced for the axial strain, the essential difference is on the radial

![Fig. 5. Variation of effective elastic compliances (initial values divided by current ones) in the triaxial compression test of 8.29 MPa confining pressure for Tennessee marble.](image)

**Table 2**

<table>
<thead>
<tr>
<th>Model parameters for a typical granite</th>
<th></th>
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<tr>
<td>( E_0 ) (MPa)</td>
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<td>( a_3 ) (MPa(^{-1}))</td>
<td>(-3.66 \times 10^{-6})</td>
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<td>( \nu_0 )</td>
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<tr>
<td>( a_4 ) (MPa(^{-1}))</td>
<td>(5.58 \times 10^{-6})</td>
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<td>( c_r ) (MPa)</td>
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<td>( t )</td>
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<td>( r_0 )</td>
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<td>( x )</td>
<td>0.26</td>
</tr>
<tr>
<td>( \beta )</td>
<td>2.5</td>
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</table>

![Fig. 6. (a) Overall strain–stress curves in the triaxial compression test for the granite with 20 MPa confining pressure (thick lines are numerical simulations). (b) Radial strain with unloading paths in the triaxial compression test for the granite with 20 MPa confining pressure (thick lines are numerical simulations). (c) Axial strain with unloading paths in the triaxial compression test for the granite with 20 MPa confining pressure (thick lines are numerical simulations).](image)
strain. Because of probable strain localization and other unstable phenomena in this zone, the physical pertinence of experimental data in this zone is questionable. Therefore, this point will be investigated in a more detailed way in our future work by associating localization analysis with the constitutive model. Numerical predictions of the effective elastic compliances are presented in Fig. 5 as functions of radial strain for the test with 8.29 MPa confining pressure. The general forms are in agreement with typical experimental data for brittle rocks. In particular, the induced anisotropy due to oriented microcracks is clearly reproduced.

The numerical simulations for the granite are presented in Figs. 6 and 7 and compared with experimental data. Again a good agreement is obtained not only for overall stress–strain curves, but also for the variation of effective elastic constants. Fig. 7 shows the simulation of a lateral extension test. In this test, the sample is first submitted to a hydrostatic stress and then the confining pressure is reduced by keeping the axial stress at a constant value. For hard rocks like the granite studied, a pre-deviatoric stress is usually necessary before the lateral stress is reduced in order to facilitate the damage initiation. This loading path is interesting because it approximates stress evolutions around a tunnel during excavation. The numerical simulations are in good agreement with experimental data. Some small scatter on the axial strain is obviously related to perturbations in experimental measurements at the beginning of the test.

### 4. Concluding remarks

A new microcrack-based continuous damage model is proposed. This model combines principal advantages of phenomenological and micromechanical models. The damage evolution law is determined from propagation condition of microcracks. The effective elastic compliance of damaged material is explicitly expressed through the suitable Gibbs free energy function. Inelastic damage strain due to residual crack opening is taken into account. The proposed model is applied to the Tennessee marble and a typical granite. A good agreement between numerical predictions and experimental data is obtained for two cases. The proposed model is able to describe the main features observed in most brittle geomaterials under compression dominated stresses. However, the validity of the model still needs to be tested for other loading paths (combined compression and torsion for instance). The extension of the present model will include the description of unloading hysteresis loops. This work will also followed by localization analysis using the bifurcation theory.

### Acknowledgements

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References


