Linear Poroelasticity

J. W. Rudnicki

Department of Civil Engineering, Northwestern University, Evanston, Illinois

Contents
11.6.1 Validity .................................. 1118
11.6.2 Formulation ............................. 1118
   11.6.2.1 Isotropic Response ............. 1121
11.6.3 Field equations ......................... 1122
   11.6.3.1 Plane Strain ................. 1123
11.6.4 Material Parameters ................. 1124
References ..................................... 1125

11.6.1 VALIDITY

Linear poroelasticity is a theory that includes the coupling between linear diffusion of a mobile species and the stress and deformation of a linear elastic porous solid. This theory has been widely applied not only to soils and rock masses infiltrated by groundwater but also to coupling of fluid flow and deformation in biological materials and diffusion of hydrogen in metals. Although departures from linear behavior are common in actual materials, linear poroelasticity is an improvement over idealizations that neglect either diffusion or deformation entirely. Because parameters describing nonlinear material behavior are difficult to obtain with accuracy, poroelasticity often offers a more practical approach than complex, nonlinear models. Furthermore, the insights gained from linear poroelasticity can be valuable for interpreting more complex models.

11.6.2 FORMULATION

The general development of linear poroelasticity was first given by Biot [1], but a revealing presentation of the constitutive relations by Rice and Cleary...

In addition to the total stress $\sigma_{ij}$ and the (infinitesimal) strain of the solid matrix $\varepsilon_{ij}$, two additional variables are needed to describe the fluid phase (or other diffusing species). One choice for these is the fluid mass per unit reference volume of porous solid $m$ and the pore fluid pressure $p$.

The latter is defined as that pressure in an imaginary reservoir of fluid connected to the material point that equilibrates any mass flow to the porous solid. If a “point” of the porous solid is regarded as a representative volume element containing differently oriented fissures, then the assignment of a single scalar pore pressure to the point assumes that the time scale of deformation is slow enough to allow pressure equilibration within this element.

Linear relations for the stress and alteration of fluid mass content from a reference value $m_0$ in terms of the strain and pore pressure take the form

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} - A_{ij}p$$

$$m - m_0 = R_{ij}\varepsilon_{ij} + Qp$$

where $L_{ijkl}, A_{ij}$, and $R_{ij}$ are arrays of factors reflecting the symmetries of $\sigma_{ij}$ and $\varepsilon_{ij}$. The first of these has the usual form of the linear elastic relation with the stress replaced by the effective stress, $\sigma_{ij} + A_{ij}p$. The fluid mass content can be written as the product of $\rho$, the density of homogeneous fluid, and $v$, the apparent volume fraction of pore space, $m = \rho v$. Linearizing this relation and substituting into Eq. 1b yields

$$v - v_0 = \frac{1}{\rho_0}R_{ij}\varepsilon_{ij} + \frac{p}{\rho_0}\left(Q - \frac{\rho_0v_0}{K_f}\right)$$

where the last term results from the linearized density change and $K_f = \rho_0p/(\rho - \rho_0)$ is the bulk modulus of the pore fluid. An additional constraint on constitutive parameters arises from the relation for changes in the Helmholtz function per unit mass $\phi(\varepsilon_{ij}, m)$ for isothermal deformation:

$$d\phi = \sigma_{ij}d\varepsilon_{ij} + \mu dm$$

where the chemical potential $\mu$ is given by

$$\mu = \int_{p_0}^{p} \frac{dp}{\rho(p)}$$

(McTigue [4] has given a formulation for linearized porothermoelasticity.) A Legendre transformation to the potential $\psi = \phi - \mu m$ yields

$$d\psi = \sigma_{ij}d\varepsilon_{ij} - vdp$$
Because $L_{ijkl} = \partial^2 \psi / \partial \varepsilon_{ij} \partial \varepsilon_{kl}$, these moduli satisfy the usual elastic symmetry with respect to interchange of the first and last pair of indices. In addition, the Maxwell relation that results from computing second derivatives of $\psi$ with respect to $\varepsilon_{ij}$ and $p$ in either order is

$$\frac{\partial \sigma_{ij}}{\partial p} = -\frac{\partial \nu}{\partial \varepsilon_{ij}} \quad (6)$$

Substituting Eqs. 1a and 2 yields

$$R_{ij} = \rho_0 A_{ij} \quad (7)$$

The remaining constitutive relation is Darcy’s law:

$$q_i = -\rho_0 \kappa_{ij} \left( \frac{\partial p}{\partial x_j} - \rho_0 g_{ij}^f \right) \quad (8)$$

where $q_i$ is the fluid mass flow rate (relative to the solid phase) per unit area, $g_{ij}^f$ is the body force per unit mass of fluid, and $\kappa_{ij}$ is a permeability tensor, required to be positive definite by the second law of thermodynamics. The permeability $\kappa_{ij}$ is more typically expressed as $k_{ij}/\eta$, where $\eta$ is the viscosity of the fluid and $k_{ij}$ is a permeability tensor with dimensions of (length)$^2$, frequently measured in units of darcies (1 darcy=$10^{-12}$ m$^2$). Alternatively, the permeability is expressed as an equivalent velocity $k = \gamma_0 \kappa$ where $\gamma_0$ is the weight density of the fluid.

If deformation occurs very slowly, fluid mass diffusion will equilibrate any alterations of pore fluid pressure. In this drained limit, the pore pressure change vanishes. Thus the moduli $L_{ijkl}$ are those appropriate for drained deformation. In the contrasting undrained limit, rapid deformation allows no time for fluid mass alteration. Setting $m = m_0$ in Eq. 1b and substituting into Eq. 1a yields

$$\sigma_{ij} = L_{ijkl}^u \varepsilon_{kl} \quad (9)$$

where

$$L_{ijkl}^u = L_{ijkl} + Q^{-1} \rho_0 A_{ij} A_{kl} \quad (10)$$

are the moduli appropriate for undrained deformation. The pore pressure induced by undrained deformation can be expressed as

$$p = -\frac{1}{3} B_{ij} \sigma_{ij} \quad (11)$$

where

$$B_{ij} = \frac{3 \rho_0 A_{mn} C_{mnij}}{Q - \rho_0 A_{mn} C_{mpq} A_{pq}} \quad (12)$$

and $C_{pqrs}$ is the inverse of $L_{ijkl}$. 

Rudnicki

1120

11.6.2.1 **Isotropic Response**

Although Cheng [5] has recently given an advantageous formulation of anisotropic poroelasticity, applications assuming isotropy are much more common because of the difficulties of obtaining a complete set of material parameters. For isotropy, the tensors $A_{ij}$, $B_{ij}$, and $\kappa_{ij}$ reduce to $a$, $B$, and $\kappa$ times the Kronecker delta, $\delta_{ij}$, respectively. The coefficients $a$ and $B$ are commonly referred to as Biot’s parameter and Skempton’s coefficient, respectively. The modulus tensor $L_{ijkl}$ assumes the usual form

$$L_{ijkl} = G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (K - 2G/3)\delta_{ij}\delta_{kl}$$

(13)

with inverse

$$C_{pqrs} = \frac{1}{2G} \left\{ \frac{1}{2} (\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) - \frac{v}{1 + v} \delta_{pq}\delta_{rs} \right\}$$

(14)

where $G$ and $K$ are the drained shear and bulk moduli, respectively; and $v$ is the Poisson’s ratio for drained response, related to $G$ and $K$ by $v = (3K - 2G)/(2(G + 3K))$. The undrained modulus tensor has the same form as Eq. 13. Evaluation of Eq. 10 indicates that the shear modulus is the same for drained and undrained deformation and that the bulk modulus for undrained deformation $K_u$ is given by

$$K_u = K + a^2 \rho_0 Q^{-1}$$

(15)

The expression for $B$, obtained from Eq. 12, is

$$B = (K - K)/aK^u$$

(16)

Thus, in the isotropic case, Eqs. 1a and 1b reduce to

$$\sigma_{ij} = 2Ge_{ij} + (K - 2G/3)\delta_{ij}e_{hk} - \alpha p\delta_{ij}$$

(17a)

$$m - m_0 = \frac{\rho_0a}{K} \left\{ \frac{1}{3} \sigma_{kk} + \frac{p}{B} \right\}$$

(17b)

and the expression for the apparent volume fraction (Eq. 2) reduces to

$$\nu - \nu_0 = \frac{\alpha}{K} \left\{ \frac{1}{3} \sigma_{kk} + \frac{1}{B} p \right\} - \frac{\nu_0 p}{K_f}$$

(18)

Insight into the meaning of $a$, $B$, and $K_u$ can be obtained by considering the special loading $\sigma_{ij} = -\sigma\delta_{ij}$, $p = \sigma$. For idealized circumstances delineated by Rice and Cleary [2] (following an observation of Reference [6]), the resulting volume strain and fractional change of pore space are equal to $-\sigma$ divided by the bulk modulus of the solid constituent $K_s$. More generally, the moduli entering these expressions, denoted here (and in Reference [2]) by $K'_0$ and $K''_0$, will differ from $K_s$. Equating these expressions to those obtained from...
Eqs. 17a, 17b and 18 yields the following relations:

\[
z = 1 - \frac{K}{K'} \quad (19a)
\]

\[
B = \frac{1/K - 1/K'}{1/K - 1/K' + \nu (1/K_f - 1/K'')}
\quad (19b)
\]

\[
K_u = \frac{1/K - 1/K' + \nu (1/K_f - 1/K'')}{(1/K - 1/K')K/K' + \nu (1/K_f - 1/K'')}
\quad (19c)
\]

The undrained response can also be expressed in terms of the undrained Poisson’s ratio, \(\nu_u\), related to \(K_u\) by

\[
\nu_u = \frac{(1 + \nu)(1 - 2\nu)zB}{3 - (1 - 2\nu)zB}
\quad (20)
\]

### 11.6.3 FIELD EQUATIONS

The relevant field equations are the usual ones of solid mechanics, equilibrium and strain displacement, and, in addition, conservation of fluid mass. Equilibrium is expressed as

\[
\sigma_{ij,i} + F_j(x, t) = 0
\quad (21)
\]

where \(F_j\) is the body force per unit volume and \((\ldots)_i\) denotes \(\partial(\ldots)/\partial x_i\). The (small) strain displacement relation is

\[
\varepsilon_{ij} = \frac{(u_{i,j} + u_{j,i})}{2}
\quad (22)
\]

Conservation of fluid mass is

\[
q_{k,k} + \partial m/\partial t = H(x, t)
\quad (23)
\]

where \(H\) is a fluid mass source.

Substitution of Eq. 22 into Eq. 17a and the result into Eq. 21 yields

\[
(K + G/3)e_{,j} + Gu_{j,ii} + F_j - \zeta p_{,j} = 0
\quad (24)
\]

where \(e = \varepsilon_{kk} = u_{i,i}\) is the dilatation. Thus gradients in pore pressure act as a distribution of body forces. Consequently, if the distribution of pore pressure is known, the displacements and stresses can be determined by superposition of the solution of Eq. 24 for a Dirac singular distribution of body force (point force). In general, however, the pore pressure field is coupled to the stress field and cannot be determined independently. Alternatively, Eq. 17 can be
used to eliminate $p$ in favor of $m$:

$$(K_u + G/3)e_j + Gu_{j_{, ii}} + F_j - \frac{(K_u - K)}{\alpha \rho_0}m_j = 0$$  \hspace{1cm} (25)$$

Substituting Darcy’s law (Eq. 8), specialized to isotropy and a uniform permeability, into Eq. 23 gives

$$-\rho_0 K p_{, kk} + \partial m/\partial t = H(x, t) - \rho_0^2 K g_{, k,k}$$  \hspace{1cm} (26)$$

The divergence of Eqs. 24 and 17b can be used to eliminate $p$ from Eq. 26. The result is a diffusion equation for the fluid mass content

$$\frac{\partial m}{\partial t} - cm_{, kk} = H(x, t) + \rho_0 K \left\{ \frac{(K_u - K)}{\alpha (K_u + 4G/3)} F_{k,k} - \rho_0 g_{, k,k}^F \right\}$$  \hspace{1cm} (27)$$

where Eqs. 16 and 17a have also been used. The diffusivity $c$ is

$$c = K \frac{(K_u - K)(K + 4G/3)}{\alpha^2(K_u + 4G/3)}$$  \hspace{1cm} (28)$$

or, as given by Rice and Cleary [2] in terms of $B$, $v$, and $v_u$,

$$c = K \left[ \frac{2G(1 - v)}{(1 - 2v)} \right] \left[ \frac{B^2(1 + v_u)^2(1 - 2v)}{9(1 - v_u)(v_u - v)} \right]$$  \hspace{1cm} (29)$$

where, as they note, the first bracket is the drained elastic modulus for one-dimensional strain and the second is unity for incompressible constituents.

### 11.6.3.1 Plane Strain

Two-dimensional solutions can provide a reasonable idealization of a variety of geomechanical problems. An important simplifying feature of the plane strain formulation, noted by Rice and Cleary [2], is that the governing equations can be expressed entirely in terms of the stress and pore pressure. For plane strain deformation in the $xy$ plane, the compatibility equation, in the absence of body forces, can be expressed as

$$\nabla^2 (\sigma_{xx} + \sigma_{yy} + 2\eta p) = 0$$  \hspace{1cm} (30)$$

and the fluid mass diffusion equation, without source terms, becomes

$$(c \nabla^2 - \partial/\partial t)[\sigma_{xx} + \sigma_{yy} + (2\eta/\mu)p] = 0$$  \hspace{1cm} (31)$$

where $\nabla^2(\ldots)$ is the two-dimensional Laplace operator and the material constants enter only in the combinations

$$\eta = 3(v_u - v)/2B(1 + v_u)(1 - v)$$  \hspace{1cm} (32)$$

$$\mu = (v_u - v)/(1 - v)$$  \hspace{1cm} (33)$$
If the boundary conditions can be expressed in terms of the stress and pore pressure, then Eqs. 30 and 31 and two of the three equilibrium equations (Eq. 21) suffice to determine these quantities.

11.6.4 MATERIAL PARAMETERS

The drained elastic constants and two additional parameters are needed to characterize a linear poroelastic solid. Ideally, the drained elastic constants would be measured on a saturated sample deformed very slowly so that no pore pressure changes are induced by fluid flow. In practice, however, the drained elastic constants are often assumed to be equal to the values obtained on dry samples. An undrained test, in which fluid exchange between the sample and the surroundings is prevented, would, in principal, suffice to determine the remaining two porous media parameters, for example, , and . Unfortunately, these and other tests to determine the porous media parameters are difficult and not yet standard. Consequently, it is frequently necessary to resort to the assumption that both and are equal to the bulk modulus of the solid constituents and then to calculate , , etc., from Eqs. 19a and 19b. Values of and determined in this way by Rice and Cleary [2] in their Table 1 for six rock types range from 0.29 to 0.34 and from 0.51 to 0.88, respectively. Values of are not given but can be calculated from the values in the table and range from 0.2 to 0.7. Values for three additional rock types listed by Detournay and Cheng [3] also fall within this range.

Limiting ranges of , , and are easily obtained from Eqs. 19a, 19b and 20. From the first, it is evident that approaches unity if the drained bulk modulus is much less than the bulk modulus of the solid constituents. If, in addition, the value of is much less than the bulk modulus of the pore fluid, so that , then also approaches unity and . These approximations are appropriate for most soils. If, on the other hand, the pore fluid is very compressible so that , then and approaches zero. In this limit and, thus, and . A further complication is that, at least for geomaterials, values in the field may be different from those determined in the laboratory because of the presence of long, narrow fissures. Such fissures tend to decrease but, if they are saturated, have little effect on and consequently, tend to increase the value of relative to .

The diffusivity (Eqs. 28 and 29) controls the time scale of fluid mass flow. The diffusivity is proportional to the permeability entering Darcy's law (Eq. 8)
but also involves a combination of moduli. Again, because large fractures and joints serve as conduits for fluid flow in situ, field values of the diffusivity are typically much larger than those measured in the laboratory. Roeloffs [7] schematically summarizes values from both laboratory and field data ranging from nearly $10^5$ to $10^{-11}$ m$^2$/s. Laboratory values for the rocks tabulated by Rice & Cleary [2] are about $10^{-2}$ to $10^{-4}$ m$^2$/s except for the Berea sandstone, for which $c = 1.6$ m$^2$/s. A variety of observations related to earthquakes suggest diffusivities in the range 0.1 to 1.0 m$^2$/s, a range that may be representative of conditions near many faults, but values an order of magnitude smaller or larger are not unusual.

REFERENCES