Eshelby Transformations, Pore Pressure and Fluid Mass Changes, and Subsidence*

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ABSTRACT: This paper is motivated by a recent analysis by Walsh (2002) of subsidence above a planar reservoir. Although the problem has been treated previously by Geertsma (1966, 1973a, b) and Segall (1992), Walsh (2002) uses a different approach. In particular, he uses a cut and weld procedure, originated by Eshelby (1957), and the reciprocal theorem of elasticity. Walsh obtains the same results as Geertsma and Segall but the connection between the two approaches is not evident. The purpose of this paper is to generalize the approach used by Walsh (2002) and to develop the connection to the approach of Geertsma and Segall.

1 INTRODUCTION

Determining the stresses, strains and surface displacements due to fluid mass or pressure changes in a subsurface reservoir is important for a variety of applications, including, hydrocarbon production and storage, aquifer management and carbon sequestration. Geertsma's (1966, 1973a,b) seminal work on this subject used a thermoelastic analogy to solve the problem for a flat tabular reservoir in an elastic half-space. Segall (1989, 1992, 1994) has extended the solution and used it to evaluate the effects of fluid extraction on the occurrence of seismicity.

Walsh (2002) has recently used a different approach to reexamine the problem of the strains and surface subsidence due to fluid withdrawal from a flat tabular reservoir considered by Geertsma and Segall. In particular, he uses a cut and weld procedure originated by Eshelby (1957) and the reciprocal theorem of elasticity. Although the approach is different, the results are identical to those of Geertsma (1966, 1973a,b) and Segall (1992). The purpose of this paper is to clarify the relation between them. Unsurprisingly, the results of both approaches can be rearranged to have the same form. The vehicle for doing so is modest generalization of the results of Eshelby (1957) for ellipsoid inclusions for elastic solids to a class of problems in poroelasticity.

2 RESULTS FROM LINEAR ELASTICITY

In linear elasticity the stress σ_{ij} is related to the infinitesimal strain ε_{kl} by $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ where the modulus tensor C_{ijkl} satisfies the symmetries $C_{ijkl} = C_{jikl} = C_{klij}$. For an isotropic material

$$\sigma_{ij} = 2G\varepsilon_{ij} + (K - 2G/3)\delta_{ij}e \tag{1}$$

where $e = \varepsilon_{kk}$ is the volume strain, G is the shear modulus, K is the bulk modulus, and δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$, if $i = j$;. $\delta_{ij} = 0$, if $i \neq j$). The strain ε_{ij} is related to the gradients of the displacement u_i by

$$\varepsilon_{ij} = (1/2)\,(u_{i,j} + u_{j,i}) \tag{2}$$

where $(...)_{,i}$ denotes $\partial(...)/\partial x_i$. For quasi-static processes, in which inertia is negligible, the stress σ_{ij} must satisfy the equilibrium equation

$$\sigma_{ij,i} + F_j(\mathbf{x}, t) = 0 \tag{3}$$

where $F(\mathbf{x}, t)$ is the body force field.

Substituting the strain displacement equation (2) into (1) and the result into equilibrium (3) yields

$$(K + 2G/3)e_{,j} + Gu_{j,kk} + F_j(\mathbf{x}, t) = 0 \tag{4}$$

The displacement at $\mathbf{x}$ due to the application of a force with components P_j at the point $\boldsymbol{\xi}$ is the solution of (4) for a singular distribution of body force given by

$$F_j^{Point}(\mathbf{x}, t) = P_j\delta_{DIRAC}(\mathbf{x} - \boldsymbol{\xi}) \tag{5}$$

where $\delta_{DIRAC}(\mathbf{x})$ is the Dirac delta function. The solution can be written in the form $u_i(\mathbf{x}) = g_{ij}(\mathbf{x} - \boldsymbol{\xi})P_j$

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where the argument of g_{ij} depends only on the difference $\mathbf{x} - \boldsymbol{\xi}$ for a homogeneous material and the elastic reciprocal theorem requires that

$$g_{ij}(\mathbf{x} - \boldsymbol{\xi}) = g_{ji}(\boldsymbol{\xi} - \mathbf{x}) \tag{6}$$

By superposition, the solution for an arbitrary distribution of body force is given by

$$u_i(\mathbf{x}) = \int_{V(\xi)} g_{ij}(\mathbf{x} - \boldsymbol{\xi}) F_j(\boldsymbol{\xi}, t) dV(\boldsymbol{\xi}) \tag{7}$$

The point force solution can also be used to generate other singular solutions corresponding to point force dipoles or moments. Consider the displacement field due to the sum of a point force **P** at $\boldsymbol{\xi} + \boldsymbol{\lambda}$ and an oppositely directed point force at $\boldsymbol{\xi}$. Taking the limit $\boldsymbol{\lambda} \to 0$, but maintaining the product $M_{kl} \to P_k \lambda_l$ finite gives

$$u_i(\mathbf{x}) = M_{k\alpha} g_{ik,\alpha} = -M_{kl} g_{ik,l} \tag{8}$$

where the Greek index $(...)_{,\alpha}$ denotes denotes differentiation with respect to ξ_α, $\partial(...)/\partial\xi_\alpha$. If the forces are arranged in pairs so that there is no net force or moment, then $M_{kl} = M_{lk}$ and (8) can be written as

$$u_i(\mathbf{x}) = (1/2) M_{\alpha\beta}(g_{i\alpha,\beta} + g_{i\beta,\alpha}) \tag{9}$$

For three perpendicular double forces, $M_{kl} = M\delta_{kl}$ and

$$u_i(\mathbf{x}) = M g_{i\alpha,\alpha} = -M g_{ik,k} \tag{10}$$

This is identical to the displacement due to a center of dilatation.

3 RESULTS FROM ESHELBY

Eshelby (1957) considered the following problem: a region of a linear elastic body enclosing a volume V undergoes a change of size and shape that, in the absence of stress, could be described by the strain field $\varepsilon^*_{kl}(\boldsymbol{\xi})$. What are the resulting fields of stress and strain within and outside of V? Rather than attacking this problem directly, Eshelby (1957) used an ingenious approach of imagined cutting and welding operations to arrive at the solution in a relatively simple form.

First, imagine cutting the region V, henceforth referred to as an inclusion, free from the elastic body. Since the body is stress-free, this can be done without introducing any tractions. Next, the region V is transformed by the application of the strain field $\varepsilon^*_{kl}(\boldsymbol{\xi})$. Because this change of size and shape is assumed to occur without stress, both the inclusion and the elastic body remain free from stress. Now restore the region V to its original size and shape by the application of a stress field

$$\sigma^*_{ij}(\boldsymbol{\xi}) = -C_{ijkl}\varepsilon^*_{kl}(\boldsymbol{\xi}) \tag{11}$$

In order for this stress field to be in equilibrium, it is necessary to simultaneously apply a body force field $F^*_j(\boldsymbol{\xi})$ that satisfies (3). Hence the required body force field is given by $F^*_j(\boldsymbol{\xi}) = -\left[C_{\alpha jlm}\varepsilon^*_{lm}(\boldsymbol{\xi})\right]_{,\alpha}$. At this point, the strain is zero everywhere and the stress vanishes outside V. Inside V, the stress is $\sigma^*_{ij}(\boldsymbol{\xi})$ and there is an extraneous body force field given by $F^*_j(\boldsymbol{\xi})$.

The unwanted body force distribution can be removed by applying its negative and using (7) to calculate the resulting displacement field:

$$u_i(\mathbf{x}) = -\int_{V(\xi)} g_{ij}\left[C_{\alpha jlm}\varepsilon^*_{lm}(\boldsymbol{\xi})\right]_{,\alpha} dV \tag{12}$$

The desired solution is the sum of the displacement, strain and stress fields resulting from (12) with those that existed before the removal of the body force field. Because the displacement is zero everywhere prior to the removal of the body force, (12) gives the displacement for the problem originally posed. The strain field can be obtained by differentiation of (12). The stress outside V can be calculated from the strain but, inside V, it is necessary to add (11). Thus, the complete stress field is

$$\sigma_{ij}(\mathbf{x}) = \begin{cases} C_{ijkl}\left[\varepsilon_{kl}(\mathbf{x}) - \varepsilon^*_{kl}(\mathbf{x})\right], \text{ inside } V \\ C_{ijkl}\varepsilon_{kl}(\mathbf{x}), \text{ outside } V \end{cases} \tag{13}$$

If the transformation strain $\varepsilon^*_{kl}(\boldsymbol{\xi})$ is non-uniform, but vanishes outside a finite region $V(\boldsymbol{\xi})$, then (12) can be written as

$$u_i(\mathbf{x}) = \int_{V(\xi)} g_{ik,\alpha}\left[C_{\alpha klm}\varepsilon^*_{lm}(\boldsymbol{\xi})\right] dV \tag{14}$$

where the difference between (12) and (14) can be eliminated by using the divergence theorem to evaluate this term on a surface just outside V where $\varepsilon^*_{lm} = 0$. Differentiating to obtain the strain yields

$$\varepsilon_{ij}(\mathbf{x}) = -\int_{V(\xi)} \frac{1}{2}(g_{il,kj} + g_{jl,ki}) C_{klmn}\varepsilon^*_{mn}(\boldsymbol{\xi}) dV(\boldsymbol{\xi}) \tag{15}$$

where $(\partial/\partial x_i) = -(\partial/\partial\xi_i)$.

If the transformation strain $\varepsilon^*_{kl}(\boldsymbol{\xi})$ is uniform, ε^T_{ij}, as in the problem originally considered by Eshelby (1957), the transformation strain (and the elastic constants, if the material is homogeneous) can be removed from the integral in (14). Using the divergence theorem on the remaining term yields

$$u_i(\mathbf{x}) = C_{\alpha klm}\varepsilon^T_{lm}\int_{S(\xi)} n_\alpha g_{ik}(\mathbf{x} - \boldsymbol{\xi}) dS(\boldsymbol{\xi}) \tag{16}$$

where $S(\boldsymbol{\xi})$ is the surface enclosing V and the n_α are components of the unit outward normal to the boundary. Differentiating (16) yields the strain

$$\varepsilon_{ij}(\mathbf{x}) = C_{klmn}\varepsilon^T_{mn}\int_{S(\xi)} n_k \frac{1}{2}\left[g_{il,j} + g_{jl,i}\right] dS \tag{17}$$

Comparing the integrand of (17) with (9) reveals that the strain can be interpreted as due to the distribution of double forces and symmetric double couples of magnitude $n_k C_{klmn}\varepsilon^T_{mn}$ over $S(\boldsymbol{\xi})$. Alternatively, the strain and displacement result from the distribution of a traction t^T_j on the surface $S(\boldsymbol{\xi})$ arising from the uniform stress $C_{kjlm}\varepsilon^T_{lm}$:

$$t^T_j = n_k C_{kjlm}\varepsilon^T_{lm} \tag{18}$$

If the elastic body is isotropic and unbounded, then the function g_{ij} is obtained from the Kelvin solution (Love 1927; Sokolnikoff 1956):

$$g_{ij} = \frac{1}{4\pi G}\left[\frac{\delta_{ij}}{r} + \frac{1}{4(1-\nu)}\frac{\partial^2 r}{\partial x_i \partial x_j}\right] \tag{19}$$

where $r = |\mathbf{x} - \boldsymbol{\xi}|$ and $\nu = (K - 2G/3)/2(K + G/3)$ is Poisson's ratio.

If, in addition, the volume $V(\xi)$ is an ellipsoid, then Eshelby (1957) calculated the integrals in (17) explicitly to show that the resulting constrained strain in the inclusion is uniform; that is, when (17) is evaluated for $\mathbf{x}$ inside V , the strain is uniform and can be expressed as

$$\varepsilon_{ij} = S_{ijkl}\varepsilon^T_{kl} \tag{20}$$

The array of factors S_{ijkl} depends only on Poisson's ratio and the geometry of the ellipsoid. General expressions for the S_{ijkl} are given by Eshelby (1957). In axes coinciding with the principal axes of the ellipsoid, the only nonzero entries are of the form S_{iijj} or S_{ijij} (no sum here on i or j) and the array is symmetric with respect to interchange of the first and last two indices, but not with respect to interchange of the first pair and last pair. Rudnicki (1977) gives expressions and tabulates results for the S_{ijkl} for oblate and prolate axisymmetric ellipsoids, Rudnicki (2002) gives extensive graphical results for oblate axisymmetric ellipsoids, and Mura (1987) gives expressions for axisymmetric and cylindrical shapes.

If the transformation strain is purely volumetric $\varepsilon^*_{lm}(\boldsymbol{\xi}) = (1/3)e^*\delta_{lm}$. If the material is isotropic, then $C_{kjlm}\varepsilon^*_{lm}(\boldsymbol{\xi}) = Ke^*\delta_{kj}$ and equations (14) and (15) become

$$u_i(\mathbf{x}) = -\int_{V(\boldsymbol{\xi})} Ke^*(\boldsymbol{\xi}) g_{ik},_k(\mathbf{x} - \boldsymbol{\xi}) dV(\boldsymbol{\xi}) \tag{21}$$

and

$$\varepsilon_{ij}(\mathbf{x}) = -\int_{V(\xi)} \frac{1}{2}(g_{ik},_{kj} + g_{jk},_{ki}) Ke^*(\boldsymbol{\xi}) dV \tag{22}$$

Comparison of (21) with (10) demonstrates that the displacement is the result of a distribution of centers of dilatation of strength $Ke^*(\boldsymbol{\xi})$ over the volume $V(\boldsymbol{\xi})$.

4 PORELASTIC FLUID-INFILTRATED SOLID

In the absence of changes in pore fluid pressure (drained response), the expression for the stress in a poroelastic solid must be identical to that in ordinary elastic solid and, hence, be given by $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ where the C_{ijkl} are identified as the drained elastic moduli. If the pore pressure p (measured relative to ambient in the formation) is non-zero then a term must be added to (1)

$$\sigma_{ij} = 2G\varepsilon_{ij} + (K - 2G/3)\delta_{ij}e - \zeta p \delta_{ij} \tag{23}$$

where ζ is a porous media constant (equal to $1 - K/K'_s$; K'_s can, under appropriate circumstances, be identified with the bulk modulus of the solid constituents (Rice and Cleary 1976)). Taking the trace of (23) yields

$$\sigma = Ke - \zeta p \tag{24}$$

where $\sigma = (1/3)\sigma_{kk}$ is the mean normal stress (positive in tension).

The alteration in fluid mass content per unit (reference) volume of porous material Δm can also be given as a linear combination of the strain and pore pressure:

$$\Delta m/\rho_0 = \zeta e + p/K' \tag{25}$$

where K' is an effective bulk modulus (Rice and Cleary 1976). Using (25) to eliminate the pore pressure from (23) yields

$$\sigma_{ij} = 2G\varepsilon_{ij} + (K_u - \frac{2}{3}G)\delta_{ij}e - \zeta K'(\Delta m/\rho_0)\delta_{ij} \tag{26}$$

where

$$K_u = K + \zeta^2 K' \tag{27}$$

If the fluid mass is constant in material elements, $\Delta m = 0$, conditions are *undrained* and, thus, K_u is the undrained bulk modulus. Comparison of (23) with (26) reveals that the shear modulus G is the same for drained and undrained conditions. Taking the trace of (26) yields

$$\sigma = K_u e - \zeta K'(\Delta m/\rho_0) \tag{28}$$

Eliminating the strain between (24) and (28) gives

$$\sigma = -(K_u/\zeta K')p + (K/\zeta)(\Delta m/\rho_0) \tag{29}$$

where (27) has also been used. For undrained conditions, the change in pore pressure is proportional to the mean normal stress

$$p = -B\sigma \tag{30}$$

where B is Skempton's coefficient and can be expressed as

$$B = \zeta K'/K_u = (K_u - K)/\zeta K_u \tag{31}$$

5 GEERTSMA - SEGALL APPROACH

The analysis of Geertsma (1966; 1973a,b) is based on the observation, previously used in thermoelasticity, that if the pore pressure distribution in a poroelastic solid is known, then the displacement and stress fields can be determined by integration of an ordinary elasticity problem without the necessity to solve a boundary value problem.

Substituting the strain displacement equation (2) into (23) and the result into equilibrium (3) with zero body force yields an equation identical to (4) with the body force F_j replaced by $-\zeta\partial p/\partial x_j$. Geertsma recognized that if the pore pressure field is known, then the solution is that of the ordinary elasticity problem with an effective body force given by $-\zeta\partial p/\partial x_j$. Thus, the solution for the poroelastic problem with zero body force and a known distribution of pore pressure is given by replacing F_j in (7) by $-\zeta\partial p/\partial x_j$:

$$u_i(\mathbf{x}) = -\int_{V(\boldsymbol{\xi})} g_{i\alpha}(\mathbf{x}-\boldsymbol{\xi})\zeta p_{,\alpha}\, dV(\boldsymbol{\xi}) \qquad (32)$$

If the pore pressure change vanishes outside V, then the same steps leading to (14) yield

$$u_i(\mathbf{x}) = \int_{V(\boldsymbol{\xi})} \zeta p(\boldsymbol{\xi},t) g_{i\alpha,\alpha}(\mathbf{x}-\boldsymbol{\xi}) dV(\boldsymbol{\xi}) \qquad (33)$$

This expression is identical to (21) if $\zeta p(\boldsymbol{\xi},t)$ is replaced by $Ke^*(\boldsymbol{\xi})$ (noting that $\partial/\partial x_k = -\partial/\partial\xi_k$). As noted by Geertsma, the term $\partial g_{ij}/\partial\xi_j$ is the displacement field caused by a center of dilatation. Thus, the solution (33) is the displacement resulting from a distribution of centers of dilatation of strength $\zeta p(\boldsymbol{\xi},t)$ over the volume V. The strains can be determined by differentiation according to (2) and the stresses by substitution into (23). Because the pore pressure change is zero outside of V, conditions there are implicitly assumed to be drained.

Using (26) instead of (23) in the procedure outlined in the preceding paragraph produces an equation identical in form to (4) with the drained bulk modulus K replaced by the undrained value K_u and the body force F_j replaced by $\zeta K'(\Delta m/\rho_0)_{,j}$. Consequently, the resulting displacement field is given by (33) as a distribution of centers of dilatation with strengths $\zeta K'(\Delta m/\rho_0)$ over V. Because the fluid mass change is presumed to vanish outside V and the undrained value of the bulk modulus appears, conditions should be undrained outside V.

6 WALSH APPROACH

Walsh (2002) has suggested using the Eshelby procedure as a step in calculating the strain and the surface subsidence due to removal of fluid from or change of pressure in a subsurface reservoir. He envisions applying Eshelby's procedure to an inclusion in a half-space and, possibly, for shapes other than ellipsoidal. The property that a uniform transformation strain results in a uniform constrained strain will not, however, apply in these cases. As shown by the calculations of Seo and Mura (1979) the stress and strain are non-uniform in an ellipsoidal inclusion subjected to uniform transformation strain in an elastic half-space. An implication of this result is that alterations of fluid mass and pore pressure are not equivalent for a reservoir in a half-space as they are for a linear poroelastic reservoir in a full space. If the reservoir is subjected to a uniform alteration of pressure, the resulting strain will be non-uniform and, hence, so will be the fluid mass alteration.

Walsh (2002) considers the inclusion to undergo a fluid mass change Δm^T in the absence of stress. Setting $\sigma = 0$ in (28) yields

$$e^T = \zeta K'(\Delta m^T/\rho_0)/K_u = B(\Delta m^T/\rho_0) \qquad (34)$$

for the transformation strain. The second equation of (34) follows from (31) and is identical to (1.b) of Walsh (2002) except that he uses Δm for the fluid mass change rather than the fluid mass change per unit volume. The corresponding pressure change is given by (29) with $\sigma = 0$:

$$p^T = (KK'/K_u)(\Delta m^T/\rho_0) \qquad (35)$$

where (31) can be used to express the coefficient in different forms and the superscript "T" indicates that this is the pore pressure change associated with the transformation strain. The pressure change can be expressed in terms of the transformation strain using (34)

$$p^T = (K/\zeta)e^T \qquad (36)$$

which is the same as Walsh's (2002) espression (1.a) with $p^T = \Delta p'_f$.

Alternatively, the pressure change in the inclusion p^T can be specified. Using (24) with $\sigma = 0$ to define e^T again yields (36). Similarly, the relation between the pressure change and the fluid mass change is given by (35). Thus, the transformation strain may be defined by either drained or undrained conditions. The remainder of the operations require, however, the specification of one or the other. If the pore fluid pressure in the reservoir is regarded as specified, then it is most convenient to assume drained conditions for the remaining operations. In this case the remaining operations induce no changes in pore pressure so that p^T is the final pore pressure in the reservoir. The resulting fluid mass change can then be computed. On the other hand, if the fluid mass change is specified, then it is most convenient to assume undrained conditions for the remaining steps so that Δm^T is the final fluid mass change. Walsh (2002) uses the latter procedure but expresses the results in terms of the pressure change.

Walsh (2002) restores the size of the inclusion assuming undrained conditions. Thus, the required traction is defined by (28) with $\Delta m = 0$,

$$n_j \sigma^{Restore} = -n_j K_u e^T \tag{37}$$

From (25), using (27), this entails the following pore fluid pressure change

$$p^{Restore} = e^T (K_u - K)/\zeta = B K_u e^T \tag{38}$$

(called $\Delta p_f''$ by Walsh (2002)). The total change in pore pressure (to this point) is the sum of (36) and (38):

$$p = p^T + p^{Restore} = e^T K_u/\zeta \tag{39}$$

(where $p = e^T K_u/\zeta$ from (39) corresponds to Walsh's (2002) p^T).

Because the inclusion has been returned to its original size, it can be reinserted into the matrix. The displacements and strains caused by removing the layer of tractions that was needed to restore the shape, (37), can be obtained by identifying these tractions with (18) in (16) and (17). Equivalently they can be calculated by inserting the transformation strain into (21) and (22). If traction removal occurs under drained conditions, there is an additional fluid mass change in the reservoir with this step. This can be evaluated from (25) with $p = 0$ once the strains caused by traction removal have been determined. If the tractions are removed assuming undrained conditions, then the undrained elastic constants (K_u instead of K) should be used and the process entails an additional pressure change. The pressure change can be evaluated from (25) with $\Delta m = 0$ or from (30) once the stresses have been determined.

Walsh (2002) assumes undrained conditions for the traction removal and uses the reciprocal theorem to calculate the strain in the reservoir and the surface displacement. Although this may offer some advantages, these quantities can be evaluated directly from the expressions discussed in the preceding paragraph. The strains are given by (22) with K_u replacing K because undrained conditions are assumed. As in Geertsma and Segall, these are simply the strains due to a uniform distribution of centers of dilatation of strength $K_u e^T$. To make contact with the analysis of Walsh (2002), consider a reservoir with a flat circular shape of radius a and thickness h lying in the plane $\xi_3 = 0$. The volume strain can be written in the form

$$e(\mathbf{x}) = K_u e^T \int_{V(\xi)} \frac{\partial}{\partial \xi_3} g_{3k},_k \, dV(\boldsymbol{\xi})$$
$$+ K_u e^T \int_{V(\xi)} (\frac{\partial}{\partial \xi_l} g_{lk},_k) dV(\boldsymbol{\xi}) \tag{40}$$

where, here, $l = 1, 2$. Integration through the thickness yields

$$e(\mathbf{x}) = K_u e^T \int_{S(\xi)} \left(g_{3k},_k^+ - g_{3k},_k^- \right) dS(\xi) \tag{41}$$
$$+ h K_u e^T \int_{S(\xi)} (\frac{\partial}{\partial \xi_1} g_{1k},_k + \frac{\partial}{\partial \xi_2} g_{2k},_k) dS(\xi)$$

where the superscripts $\pm$ indicate evaluation at $\xi_3 = \pm h/2$. Because of the thinness of the reservoir, $h \ll a$, the integrand of the second term is evaluated at $\xi_3 = 0$. Using the divergence theorem in the plane $\xi_3 = 0$ reveals that this term is simply the radial displacement due to centers of dilatation located on the perimeter of the reservoir. The integrand of the first term is the difference in the x_3 displacements of centers of dilatation at $\xi_3 = \pm h/2$ (denoted $w_D^{\pm}$ in (Walsh 2002)). Walsh (2002) evaluates this expression explicitly using the solution for a center of dilatation in a half-space given by Okada (1992). In doing so, he finds that the terms arising from the presence of the free surface are negligible if the thickness and extent of the reservoir are much less than the distance to the free surface and obtains the result

$$e(\mathbf{x}) = [(1 + \nu_u)/3(1 - \nu_u)]\, e^T \tag{42}$$

If the terms arising from the presence of the free surface are negligible, the g_{ij} are those for the Kelvin solution in the infinite space and the result (42) can be established more generally and directly. Substituting (19) into (22) for uniform $e^* = e^T$ and K_u replacing K yields

$$\varepsilon_{ij}(\mathbf{x}) = -\frac{1}{4\pi} \frac{(1 + \nu_u)}{3(1 - \nu_u)} e^T \frac{\partial^2}{\partial x_i \partial x_j} \int_{V(\boldsymbol{\xi})} (\frac{1}{r}) dV(\boldsymbol{\xi}) \tag{43}$$

Eshelby (1957) cites Nabarro (1940) as attributing this result to Crum and notes that the integral is the ordinary Newtonian potential for unit density filling the volume $V(\boldsymbol{\xi})$. Consequently, when the volume strain is evaluated from (43), the derivative operator becomes the Laplacian. The result of its operation on the integral is -4π for $\mathbf{x}$ inside $V(\boldsymbol{\xi})$ and zero outside. Thus, the constrained volume strain is uniform and given by (42) independently of the shape of the reservoir.

Walsh (2002) obtains a different combination of the vertical (ε_{33}) and lateral ($\varepsilon_{11} = \varepsilon_{22}$) strains by again using the reciprocal theorem and the solution for a uniaxial strain source. The vertical strain can, however, be evaluated directly from (22)

$$\varepsilon_{33}(\mathbf{x}) = -K_u e^T \int_{V(\xi)} g_{3k},_{k3} (\mathbf{x} - \boldsymbol{\xi}) dV(\xi) \tag{44}$$

This term is identical to the first term of (40) and, hence, can be written as the first term in (41). Evaluation yields the same result obtained by Walsh (2002): ε_{33} is identical to e from (42) and, hence, the lateral strain $\varepsilon_{11} = \varepsilon_{22} = 0$. These results are consistent with those of Eshelby (1957) in the limit of a flat ellipsoidal inclusion and suggest, unsurprisingly, that the result is independent of the actual shape of the inclusion or reservoir as long as the limit is a planar surface. For a uniform volumetric transformation strain and an ellipsoidal shape, (20) reduces to $\varepsilon_{ij} = (1/3)S_{ijkk}e^T$ In the limit of a flat axisymmetric ellipsoid perpendicular to the x_3 direction the only non-zero value is $S_{33kk} = (1+\nu_u)/\,(1-\nu_u)$ (Rudnicki 2002). Consequently, ε_{33} is identical to the volume strain (42) and the lateral strains are zero ($\varepsilon_{11} = \varepsilon_{22} = 0$).

The vertical displacement of the surface can be evaluated from (14)

$$u_3(\mathbf{x}) = K_u e^T \int_{V(\boldsymbol{\xi})} \frac{\partial}{\partial \xi_k} g_{3k}(\mathbf{x}-\xi)dV \qquad (45)$$

By using the constraint on the form of the g_{ij} from the reciprocal theorem (6) and manipulations similar to those leading from (40) to (41) yield an expression is the same as that obtained by Walsh (2002) using the reciprocal theorem directly.

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