Localization in Undrained Deformation[1]

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ABSTRACT

Exploiting form invariance of a class of elastic – plastic constitutive relations for drained and undrained deformation makes it possible to compare predictions for localization under these limiting conditions. Results are shown to depend strongly on poroelastic constants reflecting the compressibility of solid and fluid constituents and parameters describing inelastic volume strain and the dependence of the yield stress in shear on mean normal stress.

INTRODUCTION

Analysis of the onset of localization (shear banding, faulting) as a bifurcation from homogeneous deformation [1, 2, 3] has proven to be a useful framework when applied to materials modeled as rate-independent. For materials that are fluid-saturated, as is often the case for geomaterials, fluid diffusion in response to deformation introduces rate-dependence even when the drained (constant pore pressure) response is rate-independent. Because only the instantaneous material properties enter the bifurcation condition, the application of this criterion to rate-dependent materials, including those for which the rate-dependence is due to fluid diffusion, is not productive and it is necessary to examine the evolution of inhomogeneities. If, however, the drained response is rate-independent, then the alternative limit of undrained deformation, corresponding to no fluid mass change in material elements, is also rate-independent.

If the undrained response is rate-independent, the bifurcation analysis can be applied. Rice [4] showed, however, that for simple shear of a dilatant, pressure-sensitive material, homogeneous undrained deformation is unstable in the sense that small spatial nonuniformities grow exponentially in time when the localization condition is met in terms of drained response. In this case, this condition is met before the localization condition is met in terms of the undrained response. Consequently, it is unlikely that homogeneous undrained deformation could be sustained beyond the point at which the localization condition is met in terms of the drained response. Although Rice [4] focused on the case of dilatant, frictional behavior, he noted that there were types of material behavior for which the condition for localization is met in terms of the undrained response before it is met for the drained response. As discussed in more detail by Rudnicki [5], these can occur when the inelastic volume deformation

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is compactant and the material is on a frictional portion of the yield surface (yield stress in shear increases with mean compressive stress) or when the inelastic volume response is dilatant but the stress state is on a cap of the yield surface (yield stress in shear decreases with increasing mean compressive stress). Such conditions can occur near a transition in yield behavior from frictional to cap-like. Applications for high porosity rocks (porosity greater than about 15%), which are typical of reservoir rocks and, hence, tend to be fluid saturated, often encounter stress states in the vicinity of such a transition.

Rice [4] also noted that his analysis was for the special deformation state of simple shear combined with uniaxial compression. For this special state the localization condition for both drained and undrained deformation occurs at a peak in the stress strain curve. Analyses for more general deformation states [1,2,3] show that, in general, localization for drained deformation is predicted to occur slightly before or well after peak.

The analysis here takes advantage of a form invariance, noted in [5], for a wide class of rate-independent elastic – plastic material models with inelastic volume change and pressure-sensitivity. In particular, the form of the constitutive relation for undrained response is identical to that for drained response. Consequently, undrained response is obtained from the drained response by substitutions that involve poroelastic material parameters. Using these same substitutions in the predicted conditions for the onset of localization and the angle of the band for drained deformation yields results for the undrained case. The results show that the predictions for undrained response depend strongly on poroelastic parameters reflecting the compressibility of the solid and fluid constituents. A limiting case is the soil mechanics approximation for which both solid and fluid constituents are much less compressible than the porous matrix.

Although, as noted earlier, a more complete analysis requires an examination of the time evolution of heterogeneities. Because of fluid diffusion in response to heterogeneous deformation, local conditions will be neither drained nor undrained regardless of the (global) conditions on the boundaries. Nevertheless, comparison of the results for the limiting cases drained and undrained response can indicate when localization is first predicted and give guidance for a fuller analysis.

DRAINED RESPONSE

The constitutive relation is assumed to be the rate-independent, incrementally linear elastic plastic constitutive relation used in [1]. Strain increments are the sum and elastic and inelastic contributions. Elastic strain increments are given by

$$d\varepsilon_{ij}^{el} = \left\{ d\sigma_{ij} - \nu/(1+\nu)\delta_{ij} d\sigma_{kk} \right\} / 2G \qquad (1)$$

where G is the shear modulus, ν is Poisson's ratio and δ_{ij} is the Kronecker delta. Inelastic strain increments have the form:

$$d\varepsilon_{ij}^{in} = \frac{1}{h}\left(\frac{s_{ij}}{2\tau} + \frac{1}{3}\beta\delta_{ij} \right)\left(\frac{s_{kl}}{2\tau} + \frac{1}{3}\mu\delta_{kl} \right) d\sigma_{kl} \qquad (2)$$

This is the most general form for a smooth yield surface and plastic potential that depend on stress only through the first and second stress invariants. In (2) $\tau = (s_{ij}s_{ij}/2)^{1/2}$ is the Mises equivalent shear stress, $s_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij}$ is the deviatoric stress, μ is a friction coefficient, and β is a dilatancy factor, equal to $d\varepsilon_{kk}^{in}/d\gamma^{in}$, where $d\gamma^{in} = (2de_{ij}^{in}de_{ij}^{in})^{1/2}$ and de_{ij}^{in} is the deviatoric portion of (2). h is a hardening modulus equal to the slope $d\tau/d\gamma^{in}$ for constant σ. The hardening modulus is related to the tangent modulus at constant σ, $h_{\tan} = d\tau/d\gamma$ by $h_{\tan} = h/(1+h/G)$ and, hence is approximately equal to it for $h/G \ll 1$. Note, however, that for common testing configurations, axisymmetric compression and extension and plane strain, σ is not constant, and, consequently, neither h nor $h_{\tan}$ is directly related to the observed stress vs. strain curve.

The analysis of [1] predicts a critical value of the hardening modulus at which localization occurs

$$\frac{h_{crit}}{G} = \frac{1}{9}\frac{1+\nu}{1-\nu}(\beta-\mu)^2 - \frac{1}{18}(1+\nu)\left[\sqrt{3}N + (\beta+\mu)\right]^2 \tag{3}$$

and an angle between the band normal and the most compressive stress direction

$$\theta_{band} = \frac{\pi}{4} + \frac{1}{2}\arcsin\alpha \tag{4}$$

where

$$\alpha = \frac{(2/3)(\beta+\mu) - N(1-2\nu)/\sqrt{3}}{\sqrt{4-N^2}} \tag{5}$$

In (3) and (5) N is equal to the intermediate principal value of the deviatoric stress, s_2, multiplied by $\sqrt{3}$ and divided by τ. The N used here is $\sqrt{3}$ times the N used in [1] and the form of (4) is given by [6]. (These expressions neglect terms of order τ/G that would enter from use of a proper co-rotational rate of stress.)

If the material is fluid saturated, then these results also apply for the limit of *drained deformation*. This limit pertains when the deformation occurs sufficiently slowly that fluid mass diffusion can alleviate any alterations in pore fluid pressure.

UNDRAINED RESPONSE

Rudnicki [5] (see, also [3, 7, 8]) has shown that the constitutive relation for undrained response has the same form as (1) and (2) with the following assumptions: (i) the elastic portion of strain increments can be described by linear, isotropic poroelasticity; (ii) the role of the pore pressure p in the inelastic strain increments is included by replacing the stress by the Terzaghi form of the effective stress, $\sigma_{ij} + p\delta_{ij}$; and (iii) the inelastic increment in the apparent void volume fraction is equal to the inelastic volume strain increment. Rice [9] has argued on theoretical grounds that (ii) and (iii) are appropriate for geomaterials in which the primary mechanisms of inelastic deformation are microcracking from sharp-tipped flaws and frictional sliding on surfaces with small real areas of contact. In addition, (ii) seems to be supported by observational evidence.

With these assumptions, the constitutive relation for the inelastic strain increments for undrained response is identical to (2) with the following replacements: (i) $(\mu,\beta)\rightarrow$ $(1-B)(\mu,\beta)$ and (ii) $h\rightarrow H=h+(\mu\beta KB/\varsigma)$. B and ζ are poroelastic parameters [10, 11]. B is Skempton's coefficient, equal to the negative of the ratio of pore pressure change to mean normal stress change for undrained elastic deformation, and $\varsigma=1-K/K_s'$ where K is the (drained) bulk modulus and K_s' is another modulus on the order of the bulk modulus of the solid constituents (ζ is the Biot parameter, often denoted α). Skempton's coefficient B and the Biot parameter ζ range from 0 to 1. In both each cases, the lower limit occurs for a very compressible pore fluid and the upper when both the fluid and solid constituents are incompressible (porous matrix is not necessarily incompressible). This upper limit is a good approximation for soils, though not generally for rocks. When $B=1$ (i) shows that the effective values of μ and β become zero for undrained deformation.

Analogous to h for drained response, the modulus H is the slope of the τ vs. γ^{in} curve at constant σ for undrained response. Note, however, that because the pore pressure is changing for undrained response, the effective compressive stress, $\sigma+p$ is not constant.

In the elastic strain increments (1), the drained Poisson's ratio ν and bulk modulus K are replaced by their undrained counterparts

$$\nu_u=\frac{3\nu+(1-2\nu)B\varsigma}{3-(1-2\nu)B\varsigma} \tag{6}$$

and K_u [10, 11]. Since the elastic shear modulus is the same for both drained and undrained deformation (in an isotropic material), the expression for K_u is the usual one in terms of shear modulus and ν_u. The undrained Poisson's ratio ranges from $\nu\le\nu_u\le 0.5$ where the upper and lower limits correspond to the conditions mentioned above. The corresponding value of K_u becomes unbounded and in this limit, undrained response is elastically incompressible. Since the effective value of β is zero for this case, both the elastic and inelastic strain increments are incompressible.

Because the undrained response has the same form as the drained response, making the same replacements in (3) and (5) yields the predictions for the band angle α_{un} and the critical (undrained) hardening modulus for localization H_{crit}:

$$\alpha_{un}=\frac{(2/3)(1-B)(\beta+\mu)-\sqrt{3}N(1-2\nu_u)}{\sqrt{4-N^2}} \tag{7}$$

and

$$\frac{H_{crit}}{G}=\frac{1}{9}\frac{1+\nu_u}{1-\nu_u}(1-B)^2(\beta-\mu)^2-\frac{1}{18}(1+\nu_u)\left[\sqrt{3}N+(1-B)(\beta+\mu)\right]^2 \tag{8}$$

The critical slope of the underlying drained deformation at which localization occurs for undrained response is $h_{crit}^{und} = H_{crit} - \mu\beta KB/\varsigma$.

Results

For incompressible solid and fluid constituents ($B = 1, \varsigma = 1, \nu_u = 0.5$), $\alpha_u = 0$ and, hence, the band angle is always 45^o, a feature noted previously by [12]. In addition, the expressions for the critical hardening moduli simplify: The terms involving β and μ vanish and the coefficient before the square bracket in (8) becomes 1/12.

Figure 1 shows an example of how the compressibility of solid and fluid constituents (reflected through values of the poroelastic constants B and ζ) affect predictions for localization for undrained conditions for $\beta = 0.3$ and $\mu = 0.6$. The horizontal axis in both panels is the deviatoric stress state parameter N which ranges from -1 for axisymmetric extension (left) to $+1$ for axisymmetric compression (right). Left panel shows the prediction for the band angle for drained deformation and for undrained deformation for three sets of B and ζ. Right panel shows the drained critical hardening modulus (3) and the undrained critical moduli, H_{crit} (8) and h_{crit}^{und} for $B = 0.8$ and $\zeta = 0.8$.

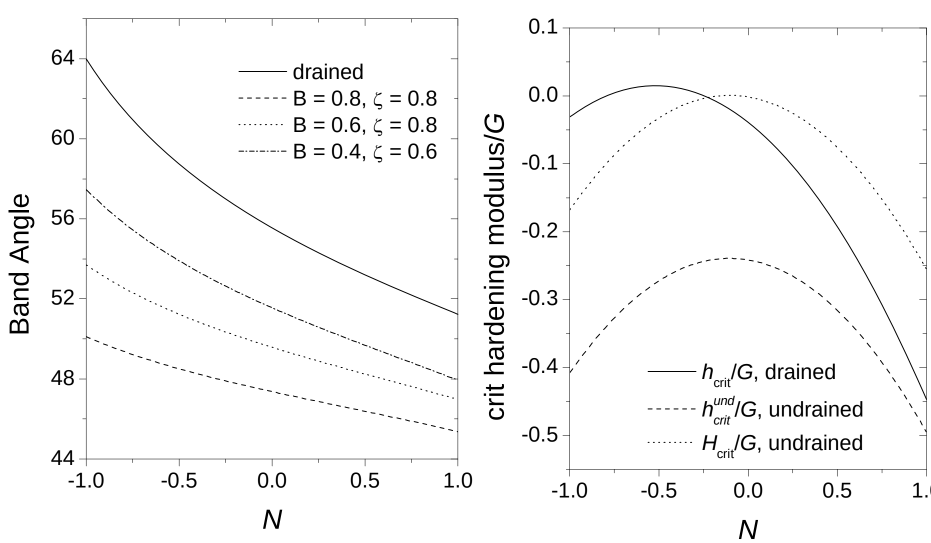

Figure 1.

The left panel shows that more compliant material constituents $(B, \zeta < 1)$ reduce the predicted band angle from the drained value, reaching 45^{o} in the limit of incompressible solid and fluid constituents (soil mechanics approximation). The right panel compares the three critical hardening moduli. Note that the critical value for undrained deformation H_{crit} can be greater or less than the drained value h_{crit}. They are, however, approximately equal and near zero slightly to the left of $N = 0$. This is consistent with both values equaling zero in the case of the simple shear case considered in [4]. Because the term $\mu\beta KB/\varsigma$ is positive if β and μ have the same sign, $h_{crit}^{und} < h_{crit}$, but the difference is smaller near axisymmetric compression $(N = 1)$ because the excess of H_{crit} over h_{crit} is larger here.

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