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Deriving Biot-Gassmann relationship by inclusion-based method

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ABSTRACT

The quasi-static theory of poroelasticity presented by Biot and Gassmann provides a relationship between the drained and undrained elastic constants of an isotropic fluid-saturated porous material in terms of the porosity of the material, bulk modulus of the solid grains, and bulk modulus of the pore fluid. We have developed an alternative approach to derive the Biot-Gassmann (BG) relationship while including the effects of the pore microstructure. First, the Eshelby transformation is used to express the local inclusion/pore strain tensor in terms of the applied strain tensor and reference material elastic properties by the superposition of a void strain and a perturbation term due to induced inclusion stress. Second, the inclusion strain expression and Hill's average principles are combined with the Mori-Tanaka/Kuster-Toksöz scheme to obtain inclusion-stress-dependent effective elastic moduli of porous materials. For an isolated pore system, the effective modulus tensor corresponds to the original Mori-Tanaka/Kuster-Toksöz's expression. Although for communicating pore system, it is proven to satisfy the BG relation. In the second case, the deformation is assumed to occur so slowly that the infiltrating fluid mass has sufficient time to diffuse between material elements and, consequently, the pore fluid pressure is equilibrated within the whole pore system. It is noteworthy that we arrive at a BG relationship without applying reciprocity theorem and that the porous material effective strain is defined from Hill's principles instead of solid phase average strain. A potential application of the stress-independent effective modulus is to help develop a dynamical modulus model of rock physics for a specific pore microstructure.

INTRODUCTION

The theory of isotropic quasi-static poroelasticity, which is first developed by Biot (1941) and Gassmann (1951) concerns averaged mechanical and hydraulic properties in fluid-infiltrated elastic solids with homogenized microscopic pores. This theory assumes that all pores are interconnected and that the pore pressure is in equilibrium in the pore space such that there is no pore fluid pressure difference between different pores. The porous frame is macroscopically and microscopically homogeneous and isotropic. Specifically, the Biot-Gassmann (BG) theory provides a relationship between the drained and undrained elastic constants of an isotropic, fluid-saturated porous material in terms of porosity of the material, elastic constants of the solid grains, and elastic constants of the pore fluid. The undrained bulk and undrained shear moduli can be expressed as

$$K_{\mathrm{ud}} = K_{\mathrm{d}} + \frac{(K_{\mathrm{s}} - K_{\mathrm{d}})^2}{(K_{\mathrm{s}} - K_{\mathrm{d}}) - \phi(K_{\mathrm{f}} - K_{\mathrm{s}})K_{\mathrm{f}}^{-1}K_{\mathrm{s}}}, \qquad (1)$$

$$G_{\mathrm{ud}} = G_{\mathrm{d}}, \qquad (2)$$

where the undrained bulk modulus K_{ud} is related to the drained bulk modulus K_{d}, bulk modulus of the solid grains K_{s}, bulk modulus of the fluid K_{f}, and the porosity of the material ϕ (the volume fraction of pores). The undrained shear modulus G_{ud} is equal to the drained shear modulus G_{d}. These two equations are commonly used in the oil and gas industry to predict the bulk modulus of rocks saturated with different fluids because the fluid compressibility affects the bulk modulus and related seismic velocities of rocks (Berge, 1998; Smith et al., 2003). Experimental data presented by Thomsen (1985) suggest that the BG relationship is indeed satisfied for several different rock types. Grechka (2009) numerically verifies that the disconnected porosity is unlikely to cause significant errors as long as the aspect ratios of pores are greater than 0.2.

However, the pore microstructure (such as shape, orientation, and size) of rocks is rarely uniform/homogeneous. A question immedi-

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ately appears that could we found the BG relation while including the effect of pore microstructure? The purposes of this paper are to present an alternative derivation of the BG relationship while emphasizing the effect of pore microstructure and to clarify that the BG relationship is satisfied as long as the pore fluid pressure is in equilibrium. Although this clarification is not completely new (e.g., Endres and Knight, 1997; Berryman, 1999; Chapman et al., 2002), the approach used here is helpful in developing micromechanics-based rock-physics models. The problems of multiphase solids and suspended grains are not considered in this study.

Many efforts have been made to rederive and extend BG theory in a variety of ways. These include Brown and Korringa (1975), Burridge and Keller (1981), Pride et al. (1992), Ciz and Shapiro (2007), Suvorov and Selvadurai (2010), and Anand (2015). For example, Brown and Korringa (1975) relax Gassmann's assumption of microscopic homogeneity and isotropy and subsequently extend the BG relationship to anisotropic materials with pore-scale heterogeneous/multiphase solids. Ciz and Shapiro (2007) further extend Brown-Korringa's relation and BG's relationship for a solid or quasi-solid (such as heavy oil) infill of pore space. Burridge and Keller (1981) extend the Biot theory for a porous elastic solid saturated by a compressible viscous fluid. When the viscosity of the fluid was small, the resulting constitutive equations became those of Biot (1941). Pride et al. (1992) derive the motion and stress-strain relations for two-phase, fluid-solid, isotropic, porous material by a direct volume averaging of the equations of motion and stress-strain relations known to apply in each phase. Their equations were shown to be consistent with Biot (1941) equations of motion and stress-strain relations. Their expressions of undrained moduli were also the same as the BG relation. Suvorov and Selvadurai (2010) develop macroscopic constitutive equations of thermoporoelasticity using the concept of eigenstrain. If the thermal effect was ignored, their constitutive equations would also be consistent with Biot's stress-strain relations. Anand (2015) notes that Biot's equations can be derived as a special case of chemoelasticity. However, in analogy to Biot (1941) and Gassmann (1951), all the rederivations and extensions stated above cannot relate the pore microstructure to overall/effective elastic moduli as well.

To concern the effect of pore microstructure on the effective elastic properties, it is common to use the Eshelby-solution-based (so-called inclusion-based) effective medium models (EMMs) that focus on the effects of microstructure of defects on macroscopic properties. Eshelby's (1957) solutions show the responses of elastic fields in an arbitrary ellipsoidal inclusion embedded in an elastic matrix material subjected to a displacement with uniform strain field at infinity. Although actual pores are seldom ellipsoidal, the ellipsoidal inclusions could cover a wide range of typical pore shape. In the presence of compliant pores (e.g., microcracks), the BG relationship was found to underestimate the wave velocities within ultrasonic frequency band (e.g., Mavko and Jizba, 1991), whereas the inclusion-based EMMs could predict the elastic moduli more closed to ultrasonic experimental data (e.g., Coyner, 1984; Murphy, 1984; Wang et al., 1991; David and Zimmerman, 2012; David et al., 2013). Physically, this is because unrelaxed pore pressures make a porous rock stiffer than it is with relaxed/homogeneous pore pressure. This phenomenon is frequently explained as a result of the so-called squirt-flow mechanism (Mavko and Jizba, 1991) that refers to the local fluid flow between compliant and stiff pores. Recent experimental studies on modulus dispersion in fluid-filled porous rocks also include Adelinet et al. (2010b), Adam and Otheim (2013), and Pimienta et al. (2015a, 2015b). Unlike the BG theory, which assumes that pores are interconnected, Eshelby's theory considers a single inclusion problem in which the pore is completely surrounded by the matrix so that no fluid mass communication is allowed for. Within inclusion-based EMMs, the pores are therefore isolated from others and unrelaxed pore fluid pressure will arise. EMMs violate BG's homogeneous pore pressure assumption in most cases except for spherical pores. However, the accepted procedure for computing BG undrained moduli is (e.g., Xu and White, 1995; Le Ravalec and Guéguen, 1996; Endres and Knight, 1997; Xu, 1998; Adelinet et al., 2010a; David and Zimmerman, 2012) estimating the drained moduli at first and then substituting the estimated drained moduli into the BG relationship to obtain the undrained moduli.

The objective of this paper was to derive the BG relationship while considering the pore microstructure effect. As stated above, the BG relationship describes the elastic properties of porous media at quasi-static condition without including pore microstructure effect. Although inclusion-based EMMs focus on pore microstructure but violate BG relationship for most pore shapes. The main idea of this study for deriving BG relationship is to extend original EMMs by assuming that pore fluid pressure is equilibrated within the whole pore system. In what follows, isolated pores refer to pores with unrelaxed fluid pressures, and communicating pores refer to undrained pores in which fluid pressures are equilibrated within the whole pore system.

To take the pore microstructure effect into account, we confine discussion to the Mori-Tanaka (MT) (Mori and Tanaka, 1973) and Kuster-Toksöz (KT) (Kuster and Toksöz, 1974) schemes. MT and KT schemes are widely used for estimating elastic properties of porous rocks by geophysics community (e.g., Kuster and Toksöz, 1974; Le Ravalec and Gueguen, 1996; Endres and Knight, 1997; Markov et al., 2005; Suvorov and Selvadurai, 2011; David and Zimmerman, 2012; Song and Hu, 2014). Compared with other typical EMMs, MT and KT models have several distinguished merits: (1) Compared with the implicit self-consistent (SC) model (e.g., Hill, 1965; Gubernatis and Kurmhansl, 1975), KT and MT models are explicit schemes and quite easy to implement. (2) Compared with the differential effective medium (DEM) model (e.g., McLaughlin, 1977; Norris, 1985), they are independent of path. (3) Compared with the explicit Eshelby's dilute model (Eshelby, 1957; Cheng, 1993; Xu, 1998), they take more matrix-inclusion and inclusion-inclusion interactions into account (Berryman and Berge, 1996). Furthermore, the effective bulk and shear moduli estimated by MT and KT models fall within rigorous upper and lower bounds (Hashin and Shtrihman, 1963) except for limiting cases with extreme microgeometries (such as high concentrations of thin disk inclusion).

This paper is organized as follows: In the "Methods of analysis" section, fundamental average principles (Hill, 1963) for stress, strain, and effective elastic properties are introduced. Eshelby's solution for strain in an elastic ellipsoidal inclusion embedded in a matrix is reformulated to be a superposition of void strain and a perturbation term due to actual stress in the inclusion. Then, the effective elastic modulus tensor is related to inclusion stress through average principles. In the "Mori-Tanaka scheme" and "Kuster-Toksöz scheme" sections, the stress-dependent effective modulus tensor is combined with MT and KT schemes to obtain the BG

relationship for the communicating pore system. The "stress-dependent effective modulus" does not mean the applied stress induced high-order (nonlinear) elastic modulus but the function of pore pressure. Finally, this paper is summarized.

METHODS OF ANALYSIS

We start this section with clarification of notations. The materials under consideration are linear elastic, isotropic materials whose stresses $\boldsymbol{\sigma}_{mn}$ and strains e_{pq} are related by the constitutive relation:

$$\boldsymbol{\sigma}_{mn} = L_{mnpq} e_{pq} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{L}\mathbf{e}. \tag{3}$$

The Einstein summary convention is used for repeated indices. The components of the fourth-order stiffness tensor L_{mnpq} are defined by

$$L_{mnpq} = K\delta_{mn}\delta_{pq} + G\left(\delta_{mp}\delta_{nq} + \delta_{np}\delta_{mq} - \frac{2}{3}\delta_{mn}\delta_{pq}\right), \tag{4}$$

where K and G are the bulk and shear moduli of the corresponding material. The indices m, n, p, and q take the values 1, 2, and 3. The function δ_{mn} is the Kronecker delta ($\delta_{mn} = 1$, if $m = n$ and $\delta_{mn} = 0$, if $m \neq n$).

Alternatively, when it is convenient, the matrix notation of tensors will be used as well. The boldface letters in equation 3 are used for this purpose. For example, $\boldsymbol{\sigma}$ and $\mathbf{e}$ represent six components vectors and $\mathbf{L}$ represents a 6×6 matrix.

To compare with the following sections, we rewrite the BG relations by matrix form:

$$\mathbf{L}_{\mathrm{ud}} - \mathbf{L}_{\mathrm{d}} = (\mathbf{L}_{\mathrm{s}} - \mathbf{L}_{\mathrm{d}})[(\mathbf{L}_{\mathrm{s}} - \mathbf{L}_{\mathrm{d}}) - \phi(\mathbf{L}_{\mathrm{f}} - \mathbf{L}_{\mathrm{s}})\mathbf{L}_{\mathrm{f}}^{-1}\mathbf{L}_{\mathrm{s}}]^{-1}(\mathbf{L}_{\mathrm{s}} - \mathbf{L}_{\mathrm{d}}), \tag{5}$$

where $\mathbf{L}_{\mathrm{d}}$ and $\mathbf{L}_{\mathrm{ud}}$ are the drained and undrained elastic modulus tensors of a porous material, respectively. The terms $\mathbf{L}_{\mathrm{s}}$ and $\mathbf{L}_{\mathrm{f}}$ are elastic modulus tensors of the solid grains and pore fluid, respectively, ϕ is the porosity. The superscript -1 denotes inverse operator. Note that the shear component in the right side of equation 5 equals to zero because the shear component of $\mathbf{L}_{\mathrm{f}}^{-1}$ tends to infinity.

Averaging methods

Fluid-saturated porous rocks are composites. Without loss of generality, for a composite material consisting of $N+1$ constituents, methods of calculating the average strain $\bar{\mathbf{e}}$ and average stress $\bar{\boldsymbol{\sigma}}$ are given by the volume averages (Hill, 1963):

$$\bar{\mathbf{e}} = \sum_{j=0}^{N} v_j \mathbf{e}_j \quad \bar{\boldsymbol{\sigma}} = \sum_{j=0}^{N} v_j \boldsymbol{\sigma}_j, \tag{6}$$

where the symbols with subscript j denote quantities of the jth constituent. The function v_j is the volume fraction of the jth constituent such that $\sum_{j=0}^{N} v_j = 1$ and $\mathbf{e}_j$ and $\boldsymbol{\sigma}_j$ are (locally averaged) strain and stress of the jth constituent. The subscript $j = 0$ defines the matrix/host constituent that commonly has the largest volume fraction. The subscript $j = 1, 2, \ldots, N$ refers to the inclusions, of which we assume there are N. Here, inclusions have distinctive elastic properties from the matrix. The constitutive relation for the jth constituent is

$$(\boldsymbol{\sigma}_j)_{mn} = (L_j)_{mnpq}(e_j)_{pq} \quad \text{or} \quad \boldsymbol{\sigma}_j = \mathbf{L}_j \mathbf{e}_j \quad (j = 0, 1, \ldots, N). \tag{7}$$

Next, for the composite, we assume that the average stress $\bar{\boldsymbol{\sigma}}$ is related to the average strain $\bar{\mathbf{e}}$ by the constitutive relation

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{L}}\,\bar{\mathbf{e}}, \tag{8}$$

where $\bar{\mathbf{L}}$ is the effective elastic modulus to be determined.

Now, equation 8 can be rewritten as

$$\begin{aligned}
\bar{\mathbf{L}}^{-1}\bar{\boldsymbol{\sigma}} = \bar{\mathbf{e}} &= \sum_{j=0}^{N} v_j \mathbf{e}_j = \sum_{j=0}^{N} v_j \mathbf{L}_j^{-1}\boldsymbol{\sigma}_j \\
&= \left(\sum_{j=1}^{N} v_j \mathbf{L}_j^{-1}\boldsymbol{\sigma}_j + v_0 \mathbf{L}_0^{-1}\boldsymbol{\sigma}_0\right) = \sum_{j=1}^{N} v_j \mathbf{L}_j^{-1}\boldsymbol{\sigma}_j + \mathbf{L}_0^{-1}(v_0\boldsymbol{\sigma}_0) \\
&= \sum_{j=1}^{N} v_j \mathbf{L}_j^{-1}\boldsymbol{\sigma}_j + \mathbf{L}_0^{-1}\left(\bar{\boldsymbol{\sigma}} - \sum_{j=1}^{N} v_j \boldsymbol{\sigma}_j\right) \\
&= \sum_{j=1}^{N} v_j (\mathbf{L}_j^{-1} - \mathbf{L}_0^{-1})\boldsymbol{\sigma}_j + \mathbf{L}_0^{-1}\bar{\boldsymbol{\sigma}} \\
&= \sum_{j=1}^{N} v_j (\mathbf{e}_j - \mathbf{L}_0^{-1}\boldsymbol{\sigma}_j) + \mathbf{L}_0^{-1}\bar{\boldsymbol{\sigma}}
\end{aligned} \tag{9}$$

which yields

$$(\mathbf{I} - \mathbf{L}_0\bar{\mathbf{L}}^{-1})\bar{\boldsymbol{\sigma}} = \sum_{j=1}^{N} v_j(\boldsymbol{\sigma}_j - \mathbf{L}_0\mathbf{e}_j) \tag{10}$$

or

$$\begin{aligned}
(\mathbf{I} - \mathbf{L}_0\bar{\mathbf{L}}^{-1})\bar{\boldsymbol{\sigma}} &= \left(\sum_{j=1}^{N} v_j\right)(\boldsymbol{\sigma}_{\mathrm{ai}} - \mathbf{L}_0\mathbf{e}_{\mathrm{ai}}) \\
&= \left(\sum_{j=1}^{N} v_j\right)(\mathbf{I} - \mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\boldsymbol{\sigma}_{\mathrm{ai}},
\end{aligned} \tag{11}$$

where $\boldsymbol{\sigma}_{\mathrm{ai}}$ and $\mathbf{e}_{\mathrm{ai}}$ denote the average stress and the average strain over all inclusions

$$\boldsymbol{\sigma}_{\mathrm{ai}} = \frac{\sum_{j=1}^{N} v_j \boldsymbol{\sigma}_j}{\sum_{j=1}^{N} v_j} \quad \mathbf{e}_{\mathrm{ai}} = \frac{\sum_{j=1}^{N} v_j \mathbf{e}_j}{\sum_{j=1}^{N} v_j} \quad \boldsymbol{\sigma}_{\mathrm{ai}} = \mathbf{L}_{\mathrm{ai}}\mathbf{e}_{\mathrm{ai}}, \tag{12}$$

where $\mathbf{I}$ is the 6×6 identity matrix and $\mathbf{L}_{\mathrm{ai}}$ is the average/effective elastic modulus for the total inclusions.

Eshelby's single-inclusion solution

To concern the microstructure effect on macroscopic properties, Eshelby's (single-inclusion) solution is recapitulated here. Eshelby

(1957) obtains the solutions of static elastic fields in an elastic ellipsoidal inclusion embedded in an elastic matrix material subjected to a displacement with uniform strain field at infinity. His solutions show that for arbitrary ellipsoidal inclusions, the resulting stress and strain fields in the inclusions are homogeneous. The inclusion strain $\mathbf{e}_{\mathrm{i}}$ can be related to the far-field applied strain $\mathbf{e}_{\infty}$ via the following equation (Wu, 1966; Qu and Cherkaoui, 2006):

$$(e_{\mathrm{i}})_{mn} = (T_{\mathrm{i}\infty})_{mnpq}(e_{\infty})_{pq} \quad \text{or} \quad \mathbf{e}_{\mathrm{i}} = \mathbf{T}_{\mathrm{i}\infty}\mathbf{e}_{\infty}, \tag{13}$$

with

$$\mathbf{T}_{\mathrm{i}\infty} = [\mathbf{I} + \mathbf{S}_{\mathrm{i}\infty}\mathbf{L}_{\infty}^{-1}(\mathbf{L}_{\mathrm{i}} - \mathbf{L}_{\infty})]^{-1}. \tag{14}$$

To differentiate from the matrix and inclusions in a finite composite, we use the subscript ∞ to denote the matrix (host/background material) of infinite extent. The symbols with the roman subscript i denote the corresponding quantities of the (dilute) inclusion. For example, $\mathbf{L}_{\infty}$ is the elastic modulus of the matrix of infinite extent, $\mathbf{L}_{\mathrm{i}}$ is the elastic modulus of the inclusion. $\mathbf{S}_{\mathrm{i}\infty}$ is the well-known Eshelby tensor, which depends on Poisson's ratio of the matrix, inclusion shape, and inclusion orientation. Expressions of $\mathbf{S}_{\mathrm{i}\infty}$ can be found in the literature (e.g., Mura, 1987; Kachanov et al., 2003; Qu and Cherkaoui, 2006). The function $\mathbf{T}_{\mathrm{i}\infty}$ is the strain transformation tensor/matrix, which is first introduced by Wu (1966).

Eshelby's theory assumes that the inclusion concentration is very dilute. This assumption implies that each inclusion is isolated from others and that the interaction between inclusions is negligible. In actual composites, the inclusion concentrations are not as diluted as Eshelby assumed, and the strain and the stress in the matrix will no longer be entirely consistent with these in Eshelby's solution. However, to take more inclusion-inclusion and inclusion-matrix interactions into account, it is convenient to modify Eshelby's solution by replacing the matrix by a reference material. The choice of reference material is different for various approximation schemes. Now, for the composite consisting of $N+1$ constituents, the strain of each constituent can be related to the strain in the reference material according to (as in Berryman and Berge (1996)

$$\mathbf{e}_j = \mathbf{T}_{j\mathrm{r}}\mathbf{e}_{\mathrm{r}} \quad (j = 0, 1, 2, \ldots, N) \tag{15}$$

with

$$\mathbf{T}_{j\mathrm{r}} = [\mathbf{I} + \mathbf{S}_{j\mathrm{r}}\mathbf{L}_{\mathrm{r}}^{-1}(\mathbf{L}_j - \mathbf{L}_{\mathrm{r}})]^{-1}, \tag{16}$$

where $\mathbf{L}_{\mathrm{r}}$ is the elastic modulus of the reference material. As defined in equation 7, $\mathbf{L}_j$ is the elastic modulus of the jth constituent. Eshelby's tensor $\mathbf{S}_{j\mathrm{r}}$ depends on Poisson's ratio of the reference material and the shape and orientation of the jth constituent.

Eshelby's solution reveals that the elastic field in the inclusion depends on externally applied loads, inclusion shape, inclusion orientation, and inclusion elastic properties. For a fluid-filled porous elastic solid, a pore with its infilling fluid is the inclusion. Shafiro and Kachanov (1997) have showed that the induced pore fluid pressures are dependent on pore shape and orientation with respect to the applied loads. The only pore shape for which no pore pressure is generated when shear stresses are applied is the sphere. Higher fluid pressure is induced in pores of small aspect ratios when a sample is under bulk compression (hydrostatic state).

When an external load is applied on a porous sample with interconnected pores, local fluid flow will take place to relax fluid pressure difference. As a result, fluid mass in a pore becomes more or less so that the effective bulk modulus in this pore differs from that in an isolated pore (e.g., Gurevich et al., 2010). The local fluid flow depends on frequency, pore microstructure, and pore conductivity. It is rather challenging to determine the effective fluid bulk modulus due to fluid communication for general cases. Nevertheless, upon the superposition law for linear elasticity problems, the pore deformation can be expressed as sum of the void deformation and a term due to the pore-fluid pressure. Without loss of generality, we can express the actual strain in the jth constituent as a superposition of a void/cavity strain $\mathbf{e}_{\mathrm{v}}$ and a perturbation term due to actual stress $\boldsymbol{\sigma}_j$. Evidently, the void must have the same size, position, and shape as the jth constituent. Rearranging equation 15 using equation 16, we have that the expression for $\mathbf{e}_j$ is (Zatsepin and Crampin, 1997)

$$\mathbf{e}_j = \mathbf{e}_{\mathrm{v}} + \mathbf{A}_{j\mathrm{r}}\boldsymbol{\sigma}_j = (\mathbf{I} - \mathbf{S}_{j\mathrm{r}})^{-1}\mathbf{e}_{\mathrm{r}} - (\mathbf{I} - \mathbf{S}_{j\mathrm{r}})^{-1}\mathbf{S}_{j\mathrm{r}}\mathbf{L}_{\mathrm{r}}^{-1}\boldsymbol{\sigma}_j, \tag{17}$$

where

$$\mathbf{e}_{\mathrm{v}} = (\mathbf{I} - \mathbf{S}_{j\mathrm{r}})^{-1}\mathbf{e}_{\mathrm{r}} \tag{18}$$

and

$$\mathbf{A}_{j\mathrm{r}} = -(\mathbf{I} - \mathbf{S}_{j\mathrm{r}})^{-1}\mathbf{S}_{j\mathrm{r}}\mathbf{L}_{\mathrm{r}}^{-1}. \tag{19}$$

Note that $\boldsymbol{\sigma}_j$ is related to $\mathbf{e}_j$ via equation 7 and $\mathbf{A}_{j\mathrm{r}}$ is independent of the elastic properties of the jth constituent. For fluid-saturated porous media, equation 17 can be used to relate the local pore-fluid pressure to macroscopic effective elastic properties.

Now, eliminating the inclusion strain tensor $\mathbf{e}_j$ $(j = 1, 2, \ldots, N)$ that appears in equation 10 using equation 17 leads to

$$(\mathbf{I} - \mathbf{L}_0\bar{\mathbf{L}}^{-1})\bar{\boldsymbol{\sigma}} = \sum_{j=1}^{N} v_j[\boldsymbol{\sigma}_j + \mathbf{L}_0(\mathbf{I} - \mathbf{S}_{j\mathrm{r}})^{-1}\mathbf{S}_{j\mathrm{r}}\mathbf{L}_{\mathrm{r}}^{-1}\boldsymbol{\sigma}_j - \mathbf{L}_0(\mathbf{I} - \mathbf{S}_{j\mathrm{r}})^{-1}\mathbf{e}_{\mathrm{r}}]. \tag{20}$$

Equation 20 connects microstructure effect to effective elastic properties. Equation 11 is a general result from average principles because no approximation has been made in arriving at it. Equation 17 relates inclusion strain to the void strain and stress perturbation. In the next section, we will show that equations 11, 17, and 20 give a unified approach for obtaining the MT and KT approximation schemes. In addition, based on these three equations, we will extend the MT and KT schemes to derive the BG relationship.

Recall that Eshelby's theory assumes that each inclusion (such as a pore) is isolated from others, whereas BG theory assumes that all pores are interconnected. The only pore geometry for which the pore stress is homogeneous and Eshelby-solution-based EMMs match the BG relationship for bulk and shear moduli is the case of a sphere. Nevertheless, to find the BG relationship using Eshelby technique, we relax Eshelby's assumption of isolation and further assume that the deformation occurs so slowly that the pore fluid pressure has sufficient time to be in equilibrium. Here, we clarify

the plausibility of doing this. First, to estimate the effective elastic properties, we use MT and KT models, which are proposed to better account for the inclusion-matrix and inclusion-inclusion interactions. The good agreement of drained effective moduli between experimental data and estimated values shows that the pore-connectedness effect on effective elastic properties is negligible, at least for drained properties (Endres and Knight, 1997; Suvorov and Selvadurai, 2011). Second, the dynamical moduli caused by pore-scale fluid flow show that the major physical factor in determining the elastic properties is the heterogeneous fluid pressure distribution (Mavko and Jizba, 1991). In fact, Eshelby's solution has been proven to be very useful in studying the pore-scale fluid flow between connected pores (squirt-flow mechanism, see O'Connell and Budiansky, 1977; Chapman et al., 2002; Pride et al., 2004; Gurevich et al., 2010; Tang et al., 2012; Song and Hu, 2013; Song et al., 2016). On the basis of these evidences, we believe that the assumption of homogeneous pore pressure can be used in MT and KT models at low frequencies.

Now, the elastic properties of reference material are unknown. The major differences in various EMMs are the distinctive approximations for the reference material and for the average strain.

MT SCHEME

Reference material and average strain

MT scheme chooses the matrix to serve as the reference material such that

$$\mathbf{L}_{\mathrm{r}} = \mathbf{L}_0 \quad \mathbf{e}_{\mathrm{r}} = \mathbf{e}_0, \tag{21}$$

and estimates the average strain by

$$\bar{\mathbf{e}} = v_0\mathbf{e}_0 + \sum_{j=1}^{N} v_j\mathbf{e}_j. \tag{22}$$

Inclusion-stress-dependent effective elastic modulus tensor

Letting r be replaced by zero and substituting equation 17 into equation 22, one gets the following expression for the average strain $\bar{\mathbf{e}}$:

$$\bar{\mathbf{e}} = \left[v_0\mathbf{I} + \sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]\mathbf{e}_0 - \sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j. \tag{23}$$

Now, inserting equation 21 into equation 20 and then eliminating $\mathbf{e}_0$ by equation 23 gives

$$\begin{aligned}(\mathbf{I}-\mathbf{L}_0(\bar{\mathbf{L}}^{\mathrm{MT}})^{-1})\bar{\boldsymbol{\sigma}} &= \sum_{j=1}^{N} v_j\boldsymbol{\sigma}_j + \sum_{j=1}^{N} v_j\mathbf{L}_0(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j \\ &- \left[\sum_{j=1}^{N} v_j\mathbf{L}_0(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]\left[v_0\mathbf{I}+\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]^{-1} \\ &\times\left[\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j\right] - \left[\sum_{j=1}^{N} v_j\mathbf{L}_0(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right] \\ &\times\left[v_0\mathbf{I}+\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]^{-1}\bar{\mathbf{e}}.\end{aligned} \tag{24}$$

The superscript MT denotes Mori-Tanaka and $\bar{\mathbf{L}}^{\mathrm{MT}}$ is the MT-estimated elastic modulus tensor. When inclusions are voids with nothing in them, substituting $\boldsymbol{\sigma}_j = 0$ into equation 24 yields

$$(\mathbf{I}-\mathbf{L}_0(\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1})\bar{\boldsymbol{\sigma}} = -\left[\sum_{j=1}^{N} v_j\mathbf{L}_0(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right] \times\left[v_0\mathbf{I}+\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]^{-1}\bar{\mathbf{e}}, \tag{25}$$

where $\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}$ is the MT estimated effective elastic modulus for a composite in which all the inclusions are replaced by voids. Applying $(\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1}\bar{\boldsymbol{\sigma}} = \bar{\mathbf{e}}$ and $\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}\bar{\mathbf{e}} = \bar{\boldsymbol{\sigma}}$, we have

$$\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}} = \mathbf{L}_0 - \mathbf{L}_0\left[\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]\left[v_0\mathbf{I}+\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]^{-1}. \tag{26}$$

Rearranging equation 26, we obtain

$$\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1} = v_0(\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1}(\mathbf{L}_0 - \bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}). \tag{27}$$

Then, substituting equation 26 into equation 24 and using $(\bar{\mathbf{L}}^{\mathrm{MT}})^{-1}\bar{\boldsymbol{\sigma}} = \bar{\mathbf{e}}$ give

$$[\mathbf{I}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}(\bar{\mathbf{L}}^{\mathrm{MT}})^{-1}]\bar{\boldsymbol{\sigma}} = \sum_{j=1}^{N} v_j\boldsymbol{\sigma}_j + \bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}\left[\sum_{j=1}^{N} v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j\right]. \tag{28}$$

Eliminating average stress $\bar{\boldsymbol{\sigma}}$ using equations 28 and 11 gives

$$(\bar{\mathbf{L}}^{\mathrm{MT}}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1}\left\{\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}\left[\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j\right]+\boldsymbol{\sigma}_{\mathrm{ai}}\sum_{j=1}^{N}v_j\right\}$$
$$=(\bar{\mathbf{L}}^{\mathrm{MT}}-\mathbf{L}_0)^{-1}(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\boldsymbol{\sigma}_{\mathrm{ai}}\sum_{j=1}^{N}v_j. \tag{29}$$

Equation 29 gives the inclusion-stress-dependent effective elastic modulus tensor.

Original MT scheme

If we substitute $\boldsymbol{\sigma}_j=\mathbf{L}_j\mathbf{T}_{j0}\mathbf{e}_0$ and equation 26 into equation 29 and use $\boldsymbol{\sigma}_{\mathrm{ai}}=\sum_{j=1}^{N}v_j\boldsymbol{\sigma}_j/\sum_{j=1}^{N}v_j$ and $\mathbf{L}_{\mathrm{ai}}^{-1}\boldsymbol{\sigma}_{\mathrm{ai}}=\mathbf{e}_{\mathrm{ai}}=\sum_{j=1}^{N}v_j\mathbf{T}_{j0}\mathbf{e}_0$, we will arrive at the original MT formulation (Mori and Tanaka, 1973; Qu and Cherkaoui, 2006):

$$\bar{\mathbf{L}}_{\mathrm{or}}^{\mathrm{MT}}=\mathbf{L}_0+\left[\sum_{j=1}^{N}v_j(\mathbf{L}_j-\mathbf{L}_0)\mathbf{T}_{j0}\right]\left(v_0\mathbf{I}+\sum_{j=1}^{N}v_j\mathbf{T}_{j0}\right)^{-1}. \tag{30}$$

The subscript or denotes original, and $\bar{\mathbf{L}}_{\mathrm{or}}^{\mathrm{MT}}$ denotes the original MT estimated effective elastic modulus tensor.

Homogeneous-stress MT scheme

If the stress is uniform/homogeneous in all inclusions, i.e., $\boldsymbol{\sigma}_j=\boldsymbol{\sigma}_{\mathrm{ai}}$ for $j=1,2,\ldots,N$, equation 29 reduces to

$$(\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1}\left\{\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}\left[\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\mathbf{L}_0^{-1}\right]+\mathbf{I}\sum_{j=1}^{N}v_j\right\}$$
$$=(\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\mathbf{L}_0)^{-1}(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\sum_{j=1}^{N}v_j. \tag{31}$$

The subscript h denotes homogeneous, and $\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}$ denotes the effective elastic modulus for the case of that the stress in all inclusions is homogeneous. Applying equation $\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\mathbf{S}_{j0}\equiv\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}-\sum_{j=1}^{N}v_j\mathbf{I}$ into equation 31, we have

$$(\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1}\left\{\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}\left[\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}-\mathbf{I}\sum_{j=1}^{N}v_j\right]\mathbf{L}_0^{-1}+\mathbf{I}\sum_{j=1}^{N}v_j\right\}$$
$$=(\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\mathbf{L}_0)^{-1}(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\sum_{j=1}^{N}v_j. \tag{32}$$

Eliminating $\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}$ by equation 27, we have

$$(\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})^{-1}(\mathbf{I}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}\mathbf{L}_0^{-1})$$
$$=(\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\mathbf{L}_0)^{-1}(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\sum_{j=1}^{N}v_j. \tag{33}$$

Rearranging equation 33, we get

$$\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}$$
$$=(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})\left[(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}})-\left(\sum_{j=1}^{N}v_j\right)(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\mathbf{L}_0\right]^{-1}(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}). \tag{34}$$

Equation 34 has the same form as the BG relationship expressed in equation 5. For fluid-saturated porous media, the subscript zero refers to the solid constituent and $\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{MT}}$ and $\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{MT}}$ represent the MT estimated drained and undrained elastic moduli, respectively. The function $\sum_{j=1}^{N}v_j$ denotes porosity, and $\mathbf{L}_{\mathrm{ai}}$ represents the pore-fluid elastic modulus.

Comparing undrained modulus with isolated pore modulus

It is of interest to compare the undrained modulus tensor with the isolated pore modulus tensor. For a porous medium saturated by a fluid, the fluid-filled pores and solid grains correspond to the inclusions and matrix, respectively. According to equation 30, the isolated pore/unrelaxed effective elastic modulus tensor $\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{MT}}$ is

$$\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{MT}}=\mathbf{L}_{\mathrm{s}}+(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})\left[\mathbf{I}+(1-\phi)\left(\sum_{j=1}^{N}\phi_j\mathbf{T}_{js}\right)^{-1}\right]^{-1}, \tag{35}$$

where

$$\mathbf{T}_{js}=[\mathbf{I}+\mathbf{S}_{js}\mathbf{L}_{\mathrm{s}}^{-1}(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})]^{-1}. \tag{36}$$

The function ϕ_j is the volume fraction of the jth pore and ϕ is the porosity. The subscript ur denotes unrelaxed pore pressure due to isolated pore structure.

According to equation 34, the MT estimated undrained modulus tensor $\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{MT}}$ can be rewritten as

$$\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{MT}}=\mathbf{L}_{\mathrm{s}}+\left[\frac{1}{\phi}\mathbf{L}_{\mathrm{s}}^{-1}\mathbf{L}_{\mathrm{f}}(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})^{-1}-(\mathbf{L}_{\mathrm{s}}-\bar{\mathbf{L}}_{\mathrm{d}}^{\mathrm{MT}})^{-1}\right]^{-1}, \tag{37}$$

where the drained modulus tensor $\bar{\mathbf{L}}_{\mathrm{d}}^{\mathrm{MT}}$ can be obtained from equation 26:

$$\bar{\mathbf{L}}_{\mathrm{d}}^{\mathrm{MT}}=\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{s}}\left[\sum_{j=1}^{N}\phi_j(\mathbf{I}-\mathbf{S}_{js})^{-1}\right]$$
$$\times\left[(1-\phi)\mathbf{I}+\sum_{j=1}^{N}\phi_j(\mathbf{I}-\mathbf{S}_{js})^{-1}\right]^{-1}. \tag{38}$$

To compare $\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{MT}}$ with $\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{MT}}$, we rewrite equation 35 using equation 36 as

$$-\frac{\phi}{1-\phi}[(\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{MT}}-\mathbf{L}_{\mathrm{s}})^{-1}-(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})^{-1}]\mathbf{L}_{\mathrm{s}} = \phi\left\{\sum_{j=1}^{N}\phi_j[(\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{f}})^{-1}\mathbf{L}_{\mathrm{f}}+(\mathbf{I}-\mathbf{S}_{j\mathrm{s}})]^{-1}\right\}^{-1} \quad (39)$$

and rewrite equation 37 using equation 38 as

$$-\frac{\phi}{1-\phi}[(\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{MT}}-\mathbf{L}_{\mathrm{s}})^{-1}-(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})^{-1}]\mathbf{L}_{\mathrm{s}} = (\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{f}})^{-1}\mathbf{L}_{\mathrm{f}}+\phi\left[\sum_{j=1}^{N}\phi_j(\mathbf{I}-\mathbf{S}_{j\mathrm{s}})^{-1}\right]^{-1}. \quad (40)$$

Subtracting equation 40 from equation 39 gives

$$\frac{\phi}{1-\phi}[(\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{MT}}-\mathbf{L}_{\mathrm{s}})^{-1}-(\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{MT}}-\mathbf{L}_{\mathrm{s}})^{-1}]\mathbf{L}_{\mathrm{s}} = \left[\sum_{j=1}^{N}\frac{\phi_j}{\phi}(\mathbf{A}+\mathbf{B}_j^{-1})^{-1}\right]^{-1}-\mathbf{A}-\left(\sum_{j=1}^{N}\frac{\phi_j}{\phi}\mathbf{B}_j\right)^{-1}, \quad (41)$$

where

$$\mathbf{A} = (\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{f}})^{-1}\mathbf{L}_{\mathrm{f}} \quad (42)$$

$$\mathbf{B}_j = (\mathbf{I}-\mathbf{S}_{j\mathrm{s}})^{-1}. \quad (43)$$

The right side of equation 41 composes a dimensionless tensor measuring modulus dispersion from undrained pores to unrelaxed pores.

KUSTER-TOKSÖZ SCHEME

Reference material and average strain

The KT scheme also chooses the matrix to serve as reference material such that

$$\mathbf{L}_{\mathrm{r}} = \mathbf{L}_0, \qquad \mathbf{e}_{\mathrm{r}} = \mathbf{e}_0, \quad (44)$$

but estimates the effective strain, in a different manner from the MT model, by

$$\bar{\mathbf{e}} = \mathbf{T}_0\mathbf{e}_0 \quad \mathbf{T}_0 = [\mathbf{I}+\mathbf{S}_0\mathbf{L}_0^{-1}(\bar{\mathbf{L}}-\mathbf{L}_0)]^{-1}, \quad (45)$$

where $\mathbf{S}_0$ is the Eshelby's tensor for spherical inclusion and depends on Poisson's ratio of the matrix material only.

Inclusion-stress-dependent effective elastic modulus tensor

Inserting equations 44 and 45 into 20 and then eliminating $\mathbf{e}_0$ give

$$\left[\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right][\mathbf{I}+(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}})^{-1}(\bar{\mathbf{L}}^{\mathrm{KT}}-\mathbf{L}_0)](\bar{\mathbf{L}}^{\mathrm{KT}})^{-1}\bar{\boldsymbol{\sigma}} = \sum_{j=1}^{N}v_j[(\mathbf{I}-\mathbf{S}_{j0})^{-1}]\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j \quad (46)$$

with

$$\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}} = \mathbf{L}_0-\mathbf{L}_0\left\{\mathbf{S}_0+\left[\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right]^{-1}\right\}^{-1}, \quad (47)$$

where the superscript KT denotes Kuster-Toksöz and $\bar{\mathbf{L}}^{\mathrm{KT}}$ is the KT-estimated effective elastic modulus tensor. Equation 46 shows that $\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}}$ is the effective elastic modulus tensor for a composite in which all the inclusions are replaced by voids.

Eliminating average stress $\bar{\boldsymbol{\sigma}}$ using equations 46 and 11 gives

$$\left[\sum_{j=1}^{N}v_j(\mathbf{I}-\mathbf{S}_{j0})^{-1}\right][(\bar{\mathbf{L}}^{\mathrm{KT}}-\mathbf{L}_0)^{-1} + (\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}})^{-1}]\left(\sum_{j=1}^{N}v_j\right)(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\boldsymbol{\sigma}_{\mathrm{ai}} = \sum_{j=1}^{N}v_j[(\mathbf{I}-\mathbf{S}_{j0})^{-1}]\mathbf{L}_0^{-1}\boldsymbol{\sigma}_j. \quad (48)$$

Equation 48 gives the inclusion-stress-dependent effective elastic modulus tensor.

Original KT scheme

If we substitute $\boldsymbol{\sigma}_j = \mathbf{L}_j\mathbf{T}_{j0}\mathbf{e}_0$ and equation 47 into equation 48 and use $\boldsymbol{\sigma}_{\mathrm{ai}} = \sum_{j=1}^{N}v_j\boldsymbol{\sigma}_j/\sum_{j=1}^{N}v_j$ and $\mathbf{L}_{\mathrm{ai}}^{-1}\boldsymbol{\sigma}_{\mathrm{ai}} = \mathbf{e}_{\mathrm{ai}} = \sum_{j=1}^{N}v_j\mathbf{T}_{j0}\mathbf{e}_0$, we will arrive at the explicit form for original KT formulation (e.g., Hu and Weng, 2000; Song and Hu, 2014):

$$\bar{\mathbf{L}}_{\mathrm{or}}^{\mathrm{KT}} = \mathbf{L}_0+\left\{\mathbf{I}-\left[\sum_{j=1}^{N}v_j(\mathbf{L}_j-\mathbf{L}_0)\mathbf{T}_{j0}\right]\mathbf{S}_0\mathbf{L}_0^{-1}\right\}^{-1} \times\left[\sum_{j=1}^{N}v_j(\mathbf{L}_j-\mathbf{L}_0)\mathbf{T}_{j0}\right]. \quad (49)$$

Homogeneous-stress KT scheme

Assuming that the stress is homogeneous in all inclusions, and thus substituting $\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_{\mathrm{ai}}$ into equation 48, we obtain

$$[(\bar{\mathbf{L}}^{\mathrm{KT}}-\mathbf{L}_0)^{-1}+(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}})^{-1}]\left(\sum_{j=1}^{N}v_j\right)(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1}) = \mathbf{L}_0^{-1}, \quad (50)$$

where $\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{KT}}$ denotes the effective elastic modulus for the case that stress in all inclusions is homogeneous. Rearranging equation 50, we have

$$\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{KT}}-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}}=(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}})\left[(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}})-\left(\sum_{j=1}^{N}v_j\right)(\mathbf{I}-\mathbf{L}_0\mathbf{L}_{\mathrm{ai}}^{-1})\mathbf{L}_0\right]^{-1}\times(\mathbf{L}_0-\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}}). \tag{51}$$

In analogy to equation 34, equation 51 has the same form as the BG relationship as well. For fluid-saturated porous media, $\bar{\mathbf{L}}_{\mathrm{v}}^{\mathrm{KT}}$ and $\bar{\mathbf{L}}_{\mathrm{h}}^{\mathrm{KT}}$ denote the KT estimated drained and undrained elastic modulus tensors, respectively. A potential application of this paper is using the stress-dependent effective modulus expressions 29 and 48 to develop an inclusion-based dynamic modulus model. The application is the subject of another paper (Song et al., 2016).

Comparing undrained modulus with isolated pore modulus

According to equation 49, the isolated pore/unrelaxed effective elastic modulus $\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{KT}}$ is

$$\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{KT}}=\mathbf{L}_{\mathrm{s}}+\left\{\left(\sum_{j=1}^{N}\phi_j\mathbf{T}_{js}\right)^{-1}(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})^{-1}-\mathbf{S}_{\mathrm{s}}\mathbf{L}_{\mathrm{s}}^{-1}\right\}^{-1}. \tag{52}$$

Remember $\mathbf{T}_{js}$ is defined in equation 36.

According to equation 51, the KT estimated undrained modulus tensor $\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{MT}}$ can be rewritten as

$$\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{KT}}=\mathbf{L}_{\mathrm{s}}+\left[\frac{1}{\phi}\mathbf{L}_{\mathrm{s}}^{-1}\mathbf{L}_{\mathrm{f}}(\mathbf{L}_{\mathrm{f}}-\mathbf{L}_{\mathrm{s}})^{-1}-(\mathbf{L}_{\mathrm{s}}-\bar{\mathbf{L}}_{\mathrm{d}}^{\mathrm{KT}})^{-1}\right]^{-1}, \tag{53}$$

where the drained modulus tensor $\bar{\mathbf{L}}_{\mathrm{d}}^{\mathrm{KT}}$ can be obtained from equation 47,

$$\bar{\mathbf{L}}_{\mathrm{d}}^{\mathrm{KT}}=\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{s}}\left\{\mathbf{S}_{\mathrm{s}}+\left[\sum_{j=1}^{N}\phi_j(\mathbf{I}-\mathbf{S}_{js})^{-1}\right]^{-1}\right\}^{-1}. \tag{54}$$

To compare $\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{KT}}$ with $\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{KT}}$, we rewrite equation 52 using equation 36 as

$$-(\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{KT}}-\mathbf{L}_{\mathrm{s}})^{-1}\mathbf{L}_{\mathrm{s}}=\left\{\sum_{j=1}^{N}\phi_j[(\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{f}})^{-1}\mathbf{L}_{\mathrm{f}}+(\mathbf{I}-\mathbf{S}_{js})]^{-1}\right\}^{-1}+\mathbf{S}_{\mathrm{s}} \tag{55}$$

and rewrite equation 53 using equation 54 as

$$-(\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{KT}}-\mathbf{L}_{\mathrm{s}})^{-1}\mathbf{L}_{\mathrm{s}}=\frac{1}{\phi}(\mathbf{L}_{\mathrm{s}}-\mathbf{L}_{\mathrm{f}})^{-1}\mathbf{L}_{\mathrm{f}}+\left[\sum_{j=1}^{N}\phi_j(\mathbf{I}-\mathbf{S}_{js})^{-1}\right]^{-1}+\mathbf{S}_{\mathrm{s}}. \tag{56}$$

Subtracting equation 56 from equation 55 gives

$$\phi[(\bar{\mathbf{L}}_{\mathrm{ud}}^{\mathrm{KT}}-\mathbf{L}_{\mathrm{s}})^{-1}-(\bar{\mathbf{L}}_{\mathrm{ur}}^{\mathrm{KT}}-\mathbf{L}_{\mathrm{s}})^{-1}]\mathbf{L}_{\mathrm{s}}=\left[\sum_{j=1}^{N}\frac{\phi_j}{\phi}(\mathbf{A}+\mathbf{B}_j^{-1})^{-1}\right]^{-1}-\mathbf{A}-\left(\sum_{j=1}^{N}\frac{\phi_j}{\phi}\mathbf{B}_j\right)^{-1}, \tag{57}$$

where $\mathbf{A}$ and $\mathbf{B}_j$ are defined in equations 42 and 43.

MODULUS DISPERSION DUE TO PORE MICROSTRUCTURE

Comparing equations 41 and 57, we find that the modulus dispersion due to pore microstructure in MT and KT schemes is controlled by the dimensionless term:

$$\mathbf{D}=\left[\sum_{j=1}^{N}\frac{\phi_j}{\phi}(\mathbf{A}+\mathbf{B}_j^{-1})^{-1}\right]^{-1}-\mathbf{A}-\left(\sum_{j=1}^{N}\frac{\phi_j}{\phi}\mathbf{B}_j\right)^{-1}. \tag{58}$$

If the porous sample contains one single pore or only one kind of pore with same orientation and shape, then $\mathbf{D}\equiv 0$ and therefore the isolated pore modulus is identical to undrained modulus. In this case, the effective fluid-saturated elastic modulus tensor is not restricted to isotropy. For aligned spheroidal pores, the effective elastic modulus tensor corresponds to transversely isotropic-extended BG relationship. In particular, for spherical pores, the isotropic isolated pore modulus is equal to undrained modulus.

In what follows, we confine discussion to isotropy. Therefore, $\mathbf{D}$ can be decomposed as

$$\mathbf{D}=P\mathbf{I}^h+Q\mathbf{I}^d=(P,Q), \tag{59}$$

where the fourth-order tensors $I^h_{ijkl}=(1/3)\delta_{ij}\delta_{kl}$ and $I^d_{ijkl}=(1/2)\delta_{ik}\delta_{jl}+(1/2)\delta_{il}\delta_{jk}-(1/3)\delta_{ij}\delta_{kl}$. The values P and Q correspond to trace and deviatoric parts of $\mathbf{D}$, respectively. Consequently, P and Q determine the bulk modulus and shear modulus dispersions, respectively.

$$\frac{\phi}{1-\phi}\left(\frac{1}{K_{\mathrm{ud}}^{\mathrm{MT}}-K_{\mathrm{s}}}-\frac{1}{K_{\mathrm{ur}}^{\mathrm{MT}}-K_{\mathrm{s}}}\right)K_{\mathrm{s}}=\phi\left(\frac{1}{K_{\mathrm{ud}}^{\mathrm{KT}}-K_{\mathrm{s}}}-\frac{1}{K_{\mathrm{ur}}^{\mathrm{KT}}-K_{\mathrm{s}}}\right)K_{\mathrm{s}}=P \tag{60}$$

$$\frac{\phi}{1-\phi}\left(\frac{1}{G_{\mathrm{ud}}^{\mathrm{MT}}-G_{\mathrm{s}}}-\frac{1}{G_{\mathrm{ur}}^{\mathrm{MT}}-G_{\mathrm{s}}}\right)G_{\mathrm{s}}=\phi\left(\frac{1}{G_{\mathrm{ud}}^{\mathrm{KT}}-G_{\mathrm{s}}}-\frac{1}{G_{\mathrm{ur}}^{\mathrm{KT}}-G_{\mathrm{s}}}\right)G_{\mathrm{s}}=Q \tag{61}$$

Equations 60 and 61 show that the difference in modulus dispersion between MT and KT schemes is negligible for small porosities. If P (or Q) is positive, $K_{\mathrm{ur}}>K_{\mathrm{ud}}$ (or $G_{\mathrm{ur}}>G_{\mathrm{ud}}$). If P (or Q) is zero, $K_{\mathrm{ur}}=K_{\mathrm{ud}}$ (or $G_{\mathrm{ur}}=G_{\mathrm{ud}}$).

For a porous sample containing N' families of randomly orientated pores, the orientation average rule (Dvorak, 2012) should be used to determine P and Q:

$$P = \left\{ \sum_{\alpha=1}^{N'} \frac{\phi_\alpha}{\phi} \frac{1}{3} [(\mathbf{A} + \mathbf{B}_\alpha^{-1})^{-1}]_{mmnn} \right\}^{-1} - \frac{K_f}{K_s - K_f} - \left[\sum_{\alpha=1}^{N'} \frac{\phi_\alpha}{\phi} \frac{1}{3} (\mathbf{B}_\alpha)_{mmnn} \right]^{-1}, \tag{62}$$

$$Q = \left(\frac{1}{5} \sum_{\alpha=1}^{N} \frac{\phi_\alpha}{\phi} \left\{ [(\mathbf{A} + \mathbf{B}_\alpha^{-1})^{-1}]_{mnmn} - \frac{1}{3} [(\mathbf{A} + \mathbf{B}_\alpha^{-1})^{-1}]_{mmnn} \right\} \right)^{-1} - \left\{ \frac{1}{5} \sum_{\alpha=1}^{N} \frac{\phi_\alpha}{\phi} [(\mathbf{B}_\alpha)_{mnmn} - \frac{1}{3} (\mathbf{B}_\alpha)_{mmnn}] \right\}^{-1}. \tag{63}$$

The terms P and Q are functions of the fluid bulk modulus, solid Poisson's ratio, and pore microstructure.

Let us consider a porous sample containing only one family of randomly oriented spheroidal pores with the same shape. Figure 1 shows how P and Q vary with pore aspect ratio (see its definition in David and Zimmerman, 2011) for different values of Poisson's ratio ν. The parameters used are the solid bulk modulus $K_s = 38$ GPa and the fluid bulk modulus $K_f = 2.2$ GPa. It is shown that P is always zero for all aspect ratios, whereas Q is positive except for the spherical pore for which $Q = 0$. This illustrates that for a porous sample containing only one family of randomly orientated pores, $K_{ur} = K_{ud}$ and $G_{ur} \geq G_{ud}$. This result is the same as that in Adelinet et al.'s (2010a) EMM. Under bulk compression (hydrostatic stress) state, the induced pore pressure in a given pore does not depend on its orientation, and all pores will have the same value of fluid pressure. In this case, the isolated pore bulk modulus is reconciled with undrained bulk modulus. Under shear stress state, the induced pore pressure within an isolated pore depends on the pore orientation (Shafiro and Kachanov, 1997). The only shape for which no pore pressure is generated when shear stresses are applied is the spherical pore.

Figure 2 plots Q versus pore aspect ratio for different values of the fluid bulk modulus. Parameters used are the solid bulk modulus $K_s = 38$ GPa and solid Poisson's ratio $\nu = 0.3$. This reveals that a smaller fluid bulk modulus leads to weaker shear modulus dispersion.

If a fluid-saturated rock is subjected to a passing stress wave, the pore pressure buildup would be dependent on frequency. Biot's (1956) theory of poroelastodynamics describes a wave-induced fluid flow resulting from wavelength-scale pressure gradients between peaks and troughs of a passing wave. However, for most crustal rocks, such a macroscopic or global flow gives rise to negligible wave attenuation, and it is not the dominant dispersion mechanism of a seismic wave.

Therefore, it is of particular interest to investigate the effect of pore structure on elastic modulus dispersion. Such a microscopic-scale effect can be modeled within the concept of squirt flow due to the presence of cracks. To obtain the modulus dispersion at intermediate frequencies, we adopt Gurevich et al.'s (2010) squirt flow model to investigate the elastic moduli in frequency domain. As to be shown that the BG relations and isolated pore (no flow) equations give the lower and upper bounds of elastic moduli in frequency domain. The squirt flow model replaces Biot's drained (dry/frame) bulk modulus by a modified frame bulk modulus, which is frequency dependent. The modified frame bulk modulus was derived in an imaginary state in which the stiff pores were dry but the cracks were filled by a fluid with a complex, frequency-dependent bulk modulus that was dependent on the aspect ratio and viscosity. By replacing Biot's drained bulk modulus by such a modified modulus in the Gassmann equation, a dynamic bulk modulus is obtained for fluid-saturated rocks. The shear dispersion can also be obtained from the bulk dispersion through a constant proportional coefficient 4/15. The modified frame bulk modulus is a function of the dry bulk modulus, the solid bulk modulus, the effective fluid bulk modulus, and the dry bulk modulus at the highest confining pressure available within the elastic regime so that all cracks are closed. Detailed equations of the squirt flow model can be found in their original paper (Gurevich et al., 2010).

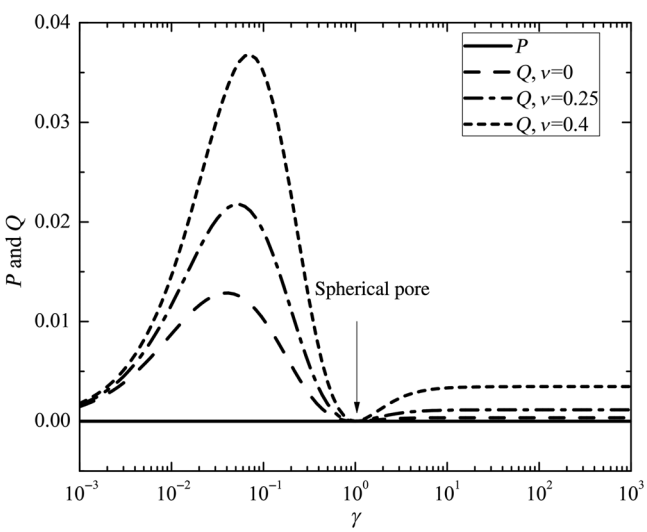

Figure 1. Dimensionless modulus dispersion as a function of the pore aspect ratio.

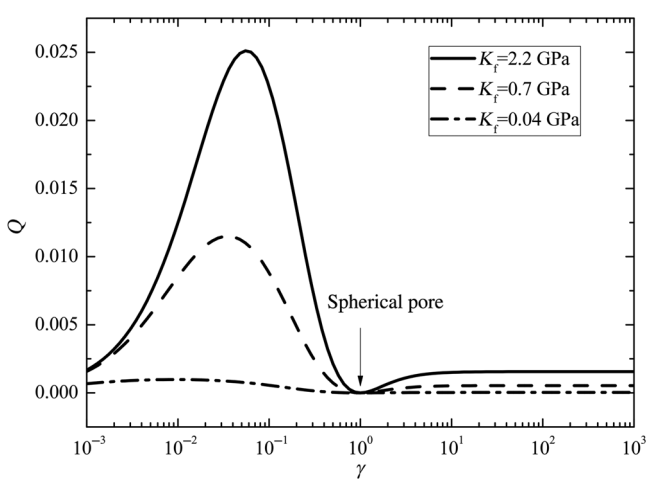

Figure 2. Dimensionless variable Q as a function of the pore aspect ratio for three values of the fluid bulk modulus.

Knowing the aspect ratio, the modified bulk modulus can also be calculated by the method of effective medium theory.

Subramaniyan et al. (2015) use Gurevich et al.'s (2010) model in conjunction with the Gaussian distributions of aspect ratios of compliant pores to calculate the squirt flow dispersion. To better interpret the squirt flow dispersion for a real rock, we use inverted data of the distributions of aspect ratios for calculation. The inverted data are for a Vosges sandstone and are from David and Zimmerman (2012).

As an example, we used the Mori-Tanaka scheme to compute the dry bulk modulus as well as its high-pressure value to obtain the modified frame bulk modulus. Then, using the equations of the squirt flow model in Gurevich et al. (2010), we obtain the effective saturated bulk modulus K_{eff} and shear modulus G_{eff}. Figures 3 and 4 plot $\mathrm{Re}(K_{\mathrm{eff}})$ and $\mathrm{Re}(G_{\mathrm{eff}})$ versus the frequency ω for the Vosges sandstone, respectively. The input data can be found in David and Zimmerman (2012). In the two figures, we also plot the BG moduli K_{ud} and G_{ud}, and no flow (or unrelaxed) moduli K_{ur} and G_{ur}. It is shown that the BG relations and no-flow equations correspond to the low- and high-frequency limits, respectively.

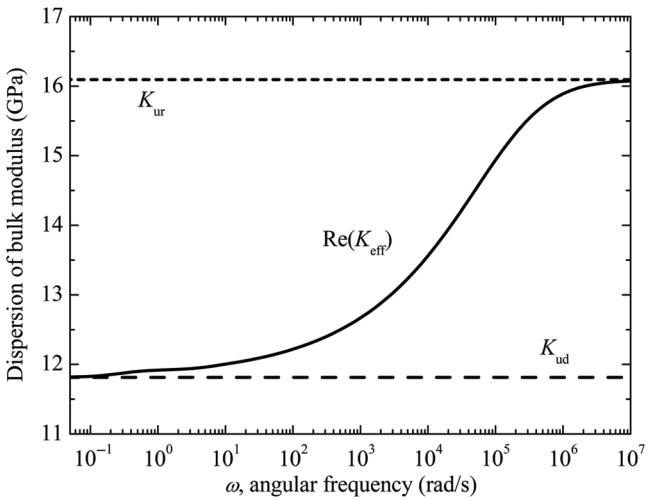

Figure 3. Squirt flow dispersion of the bulk modulus of the fluid-saturated Vosges sandstone.

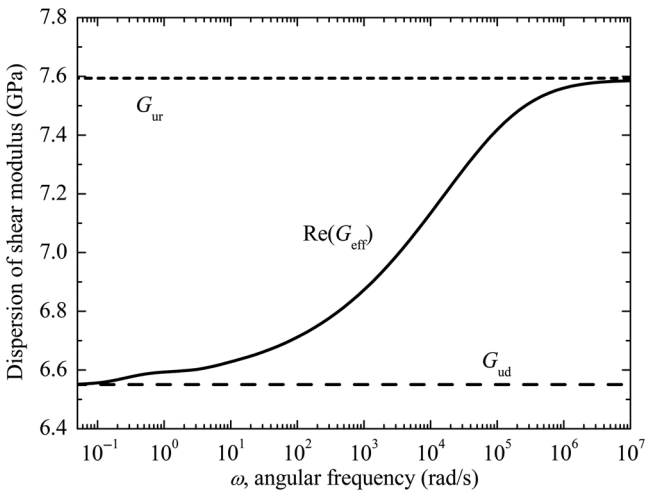

Figure 4. Squirt flow dispersion of the shear modulus of the fluid-saturated Vosges sandstone.

CONCLUSION

This paper presents an alternative derivation of BG relationship while accounting for the microstructural effect. It highlights that the BG relationship holds as long as the pore fluid pressure is homogeneous in all pore space even if the pore microstructure effect is included. The derivation is based on the extending two commonly used Eshelby-inclusion-based effective medium schemes (Mori-Tanaka and Kuster-Toksöz schemes) under communicating pore assumption. Compared with previous studies in which the porous material effective strain is defined by solid phase-average strain, this study defines the porous material effective/average strain according to Hill's average principles; i.e., the effective strain is identical to the average volume of matrix and inclusion strains. Furthermore, this study finds BG relationship without using reciprocity theorem.

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