Dynamic transverse shear modulus for a heterogeneous fluid-filled porous solid containing cylindrical inclusions

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SUMMARY
An exact analytical solution is presented for the effective dynamic transverse shear modulus in a heterogeneous fluid-filled porous solid containing cylindrical inclusions. The complex and frequency-dependent properties of the dynamic shear modulus are caused by the physical mechanism of mesoscopic-scale wave-induced fluid flow whose scale is smaller than wavelength but larger than the size of pores. Our model consists of three phases: a long cylindrical inclusion, a cylindrical shell of poroelastic matrix material with different mechanical and/or hydraulic properties than the inclusion and an outer region of effective homogeneous medium of laterally infinite extent. The behavior of both the inclusion and the matrix is described by Biot’s consolidation equations, whereas the surrounding effective medium which is used to describe the effective transverse shear properties of the inner poroelastic composite is assumed to be a viscoelastic solid whose complex transverse shear modulus needs to be determined. The determined effective transverse shear modulus is used to quantify the $S$-wave attenuation and velocity dispersion in heterogeneous fluid-filled poroelastic rocks. The calculation shows the relaxation frequency and relative position of various fluid saturation dispersion curves predicted by this study exhibit very good agreement with those of a previous 2-D finite-element simulation. For the double-porosity model (inclusions having a different solid frame than the matrix but the same pore fluid as the matrix) the effective shear modulus also exhibits a size-dependent characteristic that the relaxation frequency moves to lower frequencies by two orders of magnitude if the radius of the cylindrical poroelastic composite increases by one order of magnitude. For the patchy-saturation model (inclusions having the same solid frame as the matrix but with a different pore fluid from the matrix), the heterogeneity in pore fluid cannot cause any attenuation in the transverse shear modulus at all. A comparison with the case of spherical inclusions illustrates that the transverse shear modulus for the cylindrical inclusion exhibits more $S$-wave attenuation than spherical inclusions.

Key words: Microstructure; Seismic attenuation; Acoustic properties.

1 INTRODUCTION
Wave-induced fluid flow (WIFF) at mesoscopic scale that is smaller than wavelength but larger than the size of pores is believed to be the major physical mechanism in causing the velocity dispersion and attenuation in seismic frequency bands ($1–10^4$ Hz; Pride et al. 2004; Müller et al. 2010). A common approach to concerning the effects of mesoscopic-scale WIFF on porous rocks is to describe the rocks as viscoelastic materials whose stress–strain relation coincides with the usual equations of elasticity but with complex strain and stress fields and complex moduli. Many analytical models have been developed to quantify the dynamic bulk modulus and frequency-dependent properties of $P$ waves. These include White (1975), White et al. (1975), Dutta & Odé (1979a,b), Berryman (1985), Norris (1993), Johnson (2001), Pride & Berryman (2003), Pride et al. (2004), Müller & Gurevich (2004), Brajanovski et al. (2005), Vogelaar & Smeulders (2007), Vogelaar et al. (2010) and so on. In contrast, exact analytical solutions for the shear problem have been rarely studied.

This is the second of two papers dedicated to obtaining the effective dynamic shear modulus due to mesoscopic-scale WIFF for heterogeneous fluid-filled porous media. In the first paper (Song et al. 2016), an exact analytical expression for the isotropic dynamic shear modulus for heterogeneous fluid-filled porous media containing spherical poroelastic inclusions was presented. This paper focuses on the effective dynamic shear properties for heterogeneous fluid-filled porous solids containing cylindrical inclusion. Specifically, the effective transverse shear modulus is studied in detail. An important application of this study is to quantify the $S$-wave attenuation and velocity

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dispersion for a 2-D (plane-strain) hydrocarbon reservoir within seismic frequencies or to estimate the effective transverse shear properties for fluid-saturated crustal rocks containing caving boreholes.

In the first paper, to obtain the effective shear modulus for heterogeneous fluid-filled porous media containing spherical poroelastic inclusions, a poroelastic three-phase model is used to specify the geometric structure. In this paper, the three-phase geometry is altered to determine the effective transverse shear modulus for a heterogeneous fluid-filled porous solid containing cylindrical inclusions. The three-phase model used in the present paper is shown in Fig. 1. It involves a long cylindrical inclusion with radius \( a \), an annulus of poroelastic matrix material with thickness of \( b-a \) and an outer region of effective homogeneous medium of laterally infinite extent. As is shown in Fig. 1, an inclusion and surrounding matrix consist of a circular unit cell. The inclusion has different physical properties from the matrix. We assume that the behaviors of both the inclusion and the matrix are described by the Biot’s quasi-static theory of poroelasticity (Biot 1941) with standard conditions of Deresiewicz & Skalak (1963) at the inclusion-matrix interface. The outer effective medium, assumed to be a viscoelastic solid whose complex transverse shear modulus needs to be determined, is used to describe the effective transverse shear properties of the inner poroelastic composite. To solve for the effective transverse shear modulus for the poroelastic composite, the following section briefly reviews the governing equations of theory of poroelasticity.

2 GOVERNING EQUATIONS OF QUASI-STATIC THEORY OF POROELASTICITY

The quasi-static theory of poroelasticity was initially established by Biot (1941). Other useful references include Rice & Cleary (1976), Rudnicki (1986) and Wang (2000). For a linear, isotropic, fluid-saturated poroelastic medium, the displacement gradient of solid frame deformation, \( u_{p,q} (= \partial u_p / \partial x_q) \), and the change in fluid volume content per unit reference volume of porous solid \( \zeta \), are related to the total stress \( \sigma_{pq} \) and pore fluid pressure \( p_f \) via constitutive equations

\[
\sigma_{pq} = \mu (u_{p,q} + u_{q,p}) + \lambda u_{k,k} \delta_{pq} - \alpha p_f \delta_{pq},
\]

\[
\zeta = \alpha [u_{k,k} + \alpha p_f / (\lambda_u - \lambda)],
\]

where \( \delta_{pq} \) is the Kronecker delta. \( \lambda \) and \( \mu \) are the Lamé moduli appropriate for drained or dry response, whereas \( \lambda_u \) is the Lamé modulus for undrained response. \( \alpha = 1 - K / K_s \) is the Biot constant, \( K (= \lambda + 2\mu/3) \) is the drained bulk modulus and \( K_s \) is the bulk modulus of the solid constituents. \( \lambda_u \) is related to \( \lambda \) by \( \lambda_u = \lambda + \alpha^2 M \), where \( M = 1 / [\phi / K_f + (\alpha - \phi) / K_s] \). \( K_f \) is the fluid bulk modulus and \( \phi \) is the porosity.

The constitutive formulation is completed by Darcy’s law which, in the absence of body force, is given by

\[
\partial w_p / \partial t = -\gamma \partial p_f / \partial x_p,
\]

where the time differentiation \( \partial w_p / \partial t \) denotes the average velocity of the pore fluid relative to the solid frame on a RVE multiplied by porosity. \( w_p \) is called as filtration displacement. \( \gamma \) is related to \( \kappa \), the permeability (with dimension of length squared), and \( \eta \), the fluid viscosity by \( \gamma = \kappa / \eta \). The Darcy’s law describes a quasi-static flow in pore space.

Substituting eq. (1) into equilibrium equation \( \sigma_{pq,q} = 0 \) and then eliminating \( p_f \) by using eq. (2) yield

\[
(\lambda_u + \mu) u_{k,k,p} + \mu u_{p,k,k} = (\lambda_u - \lambda) \alpha^{-1} \zeta_p.
\]

Eq. (4) shows that a porous material behaves as an undrained medium with the gradient of change in fluid mass acting as a body force.
Substituting Darcy’s law (3) into $\dot{w}_{\perp} + \partial \zeta / \partial t = 0$, the equation of fluid mass conservation without fluid mass source, and using eq. (2) and the divergence of eq. (4) lead to a diffusion equation for $\zeta$:

$$\partial \zeta / \partial t = c \nabla^2 \zeta,$$

(5)

where the diffusivity is

$$c = \frac{\gamma (\lambda_w - \lambda) (\lambda + 2\mu)}{\alpha^2 (\alpha + 2\mu)},$$

(6)

and $\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R}.$

In the following section we will use the governing eqs (1)–(5) to derive solutions for the effective viscoelastic (complex and frequency-dependent) transverse shear modulus in the cylindrical poroelastic three-phase model.

3 EFFECTIVE TRANSVERSE SHEAR MODULUS

3.1 Expressions of displacement, stress and pore pressure

The study is restricted to a 2-D (plane-strain) problem. Fig. 2 shows a polar coordinate system with the origin at the center of the unit cell shown in Fig. 1. The two radii in Fig. 1 are taken such that $a^2 / b^2 = \beta$, the volume fraction of the inclusion. The mechanical and hydraulic properties of the inclusion and the matrix are described by Biot’s poroelastic equations. To determine the shear properties in such a poroelastic composite, let the three-phase specimen be subjected to pure shear loadings in $x$-$y$ plane at infinity with tension in $x$ direction and compression in $y$ direction. Fig. 2 shows a schematic diagram of displacement components and the pure shear loadings in the polar coordinate system. Suitable displacement components and $\zeta$ have the form

$$u_R = U_R \cos 2\vartheta,$$

(7)

$$u_\vartheta = U_\vartheta \sin 2\vartheta,$$

(8)

$$\zeta = Z \cos 2\vartheta,$$

(9)

where $U_R$, $U_\vartheta$ and $Z$ are functions of $R$ only. $R$ and $\vartheta$ are the polar coordinates shown in Fig. 2. The forms of displacement in (7) and (8) have been used by Christensen & Lo (1979) to calculate the effective transverse shear modulus in a purely elastic composite. With the expectation that the displacement solutions will involve the same polar forms which appeared in the corresponding problem of classic elasticity, eqs (7) and (8) are also used to describe shear displacements in a poroelastic medium. Moreover, since considerations of symmetry and linearity also indicate that $\zeta$ has the same form with respect to angle coordinates as radial displacement $u_R$, eq. (9) accounts for the coupled poroelastic effect. Based on the uniqueness theorem in poroelasticity (Deresiewicz & Skalak 1963), the forms of eqs (7)–(9) will be suitable as long as one is able to succeed in finding the solutions for this shear problem.
Adopting an $e^{-\omega t}$ dependence and substituting eq. (9) into the fluid diffusion eq. (5) give the general form of $Z$ (Appendix A)

$$Z(R) = EJ_2(kR) + FY_2(kR),$$  \hspace{1cm} (10)

where $\partial/\partial t = -i\omega$ is used, $J_2$ and $Y_2$ are second-order Bessel functions of two kinds, $k = \sqrt{\omega\mu/c}$ is the wave number of the Biot slow compressional wave. As illustrated in Chandler & Johnson (1981), the Biot slow compressional wave becomes a diffusive phenomenon at low frequencies, and its wavenumber can be expressed as a function of frequency and diffusivity. $\omega$ is the angular frequency. The frequency $f$ is related to $\omega$ via relation $\omega = 2\pi f$.

The general expression for displacement components $U_R$ and $U_\theta$, which are obtained by solving eq. (4) by substituting eq. (10) into (4), are (see the derivation in Appendix A)

$$U_R = AR + BR^3 + \frac{C}{R} + \frac{D}{R^3} + \frac{\lambda_u - \lambda}{\alpha H} \frac{1}{2k} E [J_0(kR) - J_1(kR)] + F [Y_0(kR) - Y_1(kR)],$$ \hspace{1cm} (11)

$$U_\theta = -AR + \left(1 - \frac{3}{2v_u}\right) BR^3 - \frac{1 - 2v_u}{2(1 - v_u)} C + \frac{D}{R^3} + \frac{\lambda_u - \lambda}{\alpha H} \frac{1}{2k} E [J_1(kR) + J_0(kR)] + F [Y_1(kR) + Y_0(kR)],$$ \hspace{1cm} (12)

where $v_u = \frac{3\mu_a - 2\mu}{3\mu_a + 2\mu}$ is the Poisson’s ratio for undrained response, $H(=\lambda_u + 2\mu = \frac{2(1-\nu)}{1-2\nu})$ is the $P$-wave modulus defined by Biot (1962). Expressions for $U_R$ and $U_\theta$ have two parts. The first part, which has terms of $R$, $R^3$, $R^{-1}$ and $R^{-3}$, with undrained Poisson’s ratio is identical to the form of solution for purely elastic material (Christensen & Lo 1979). The second part, which consists of Bessel functions, is caused by fluid flow (diffusion). Now substituting eqs (10)–(12) into eqs (7)–(9), and then substituting the results into constitutive equations yield the expressions of stresses, pore pressure and filtration displacement

$$\sigma_{RR} = \Sigma_\delta(R) \cos 2\theta,$$  \hspace{1cm} (13)

$$\sigma_{R\theta} = \Sigma_\delta(R) \sin 2\theta,$$  \hspace{1cm} (14)

$$p_f = P_f(R) \cos 2\theta,$$  \hspace{1cm} (15)

$$w_R = W_R(R) \cos 2\theta,$$  \hspace{1cm} (16)

where

$$\Sigma_\delta(R) = 2\mu_A - \frac{2\mu}{1 - v_u} C - 6\mu \frac{D}{R^3} - \mu \frac{\lambda_u - \lambda}{\alpha H} E \left[\frac{1}{2} J_0(kR) + J_2(kR) + \frac{1}{2} J_4(kR)\right] - \mu \frac{\lambda_u - \lambda}{\alpha H} F \left[\frac{1}{2} Y_0(kR) + Y_2(kR) + \frac{1}{2} Y_4(kR)\right],$$ \hspace{1cm} (17)

$$P_f(R) = \frac{\lambda_u - \lambda}{\alpha H} \left[- \left(6 - \frac{3}{v_u}\right) BR^2 + \frac{1 - 2v_u}{1 - v_u} C + \frac{\lambda_u - \lambda}{\alpha^2} \frac{H_u}{H} \left[E J_0(kR) + F Y_0(kR)\right]\right],$$ \hspace{1cm} (19)

$$W_R(R) = \frac{\gamma}{2i\omega} \frac{\lambda_u - \lambda}{\alpha} \left[\frac{6}{v_u} (1 - 2v_u) BR - \frac{2(1 - 2v_u)}{1 - v_u} C\right] + \frac{\gamma}{2i\omega} \frac{\lambda_u - \lambda}{\alpha^2} \frac{H_u}{H} \left[E J_0(kR) - J_2(kR)\right] + F [Y_1(kR) - Y_3(kR)],$$ \hspace{1cm} (20)

and where $H_u(=\lambda + 2\mu)$ is the drained $P$-wave modulus. $\sigma_{RR}$ is the normal stress in the radial direction. $\sigma_{R\theta}$ is the shear stress. $w_R$ is the radial filtration displacement.

### 3.2 Exact analytical solution

As seen in Fig. 1, there are three separate regions in which the solutions must be known. For the later use, subscripts $i$, $m$ and $e$ denote the inclusion phase, the matrix phase and the effective medium phase, respectively. Because the poroelastic solutions at $R = 0$ must be finite, this require $C_i = D_i = F_i = 0$ in the inclusion. Moreover, because the effective elastic or viscoelastic solutions at $R = \infty$ must be finite, $B_e = E_e = F_e = 0$ in the outer effective medium region. Furthermore, for the three-phase model shown in Fig. 1, Christensen & Lo (1979) give that

$$C_e = 0,$$  \hspace{1cm} (21)
by the approach of Eshelby interaction energy. Although they obtain this result in a purely elastic composite, it is appropriately used in this present outer equivalent homogeneous viscoelastic medium since the stress–strain relation of the outer viscoelastic medium is mathematically equivalent to that of an ordinary elastic medium.

Now, the general forms of solutions in the three separate regions of Fig. 1 are

$$U_{\eta i} = A_i R + B_i R^3 + \frac{\lambda_{\eta i} - \lambda_i}{\alpha_i H_i} \frac{1}{2k_i} E_i [J_3 (k, R) - J_1 (k, R)],$$  \hfill (22)

$$U_{p\mu} = -A_i R + \left(1 - \frac{3}{2 \nu_{\mu i}}\right) B_i R^3 + \frac{\lambda_{\eta i} - \lambda_i}{\alpha_i H_i} \frac{1}{2k_i} E_i [J_1 (k, R) + J_3 (k, R)].$$  \hfill (23)

$$\Sigma_{\eta i} = 2 \mu_i A_i - \mu_i \frac{\lambda_{\eta i} - \lambda_i}{\alpha_i H_i} E_i \left[\frac{1}{2} J_0 (k, R) + J_2 (k, R) + \frac{1}{2} J_4 (k, R)\right],$$  \hfill (24)

$$\Sigma_{\eta i} = -2 \mu_i A_i - \frac{3 \mu_i}{\nu_{\mu i}} B_i R^2 + \mu_i \frac{\lambda_{\eta i} - \lambda_i}{\alpha_i H_i} E_i \left[\frac{1}{2} J_0 (k, R) - \frac{1}{2} J_4 (k, R)\right],$$  \hfill (25)

$$P_{p\mu} = -\frac{\lambda_{\eta i} + \lambda_i}{\alpha_i} \left(6 - \frac{3}{2 \nu_{\mu i}}\right) B_i R^3 + \frac{\lambda_{\eta i} + \lambda_i}{\alpha_i H_i} \frac{1}{2k_i} E_i J_2 (k, R),$$  \hfill (26)

$$W_{\eta i} = \frac{\nu_{\mu i}}{\nu_{\alpha i}} \left[1 - 2 \nu_{\mu i}\right] B_i R^3 + \frac{\nu_{\mu i}}{\nu_{\alpha i}} \frac{1}{1 - \nu_{\alpha i}} E_i \left[\frac{1}{2} Y_0 (k, R) + Y_2 (k, R) - \frac{1}{2} Y_4 (k, R)\right],$$  \hfill (27)

in the inclusion phase,

$$U_{\eta m} = A_m R + B_m R^3 + \frac{C_m}{R} + D_m \frac{\lambda_{\eta m} - \lambda_m}{\alpha_m H_m} \frac{1}{2k_m} E_m \left[E_m (J_3 (k, R) - J_1 (k, R)) + F_m (Y_3 (k, R) - Y_1 (k, R))\right],$$  \hfill (28)

$$U_{p\mu m} = -A_m R + \left(1 - \frac{3}{2 \nu_{\mu m}}\right) B_m R^3 + \frac{\mu_m C_m}{\nu_{\mu m} R^3} + \frac{\mu_m D_m}{R^4} + \mu_m \frac{\lambda_{\eta m} - \lambda_m}{\alpha_m H_m} \frac{1}{2k_m} E_m \left[E_m (J_1 (k, R) + J_3 (k, R)) + F_m (Y_1 (k, R) + Y_3 (k, R))\right],$$  \hfill (29)

$$\Sigma_{\eta m} = 2 \mu_m A_m - \frac{2 \mu_m C_m}{1 - \nu_{\mu m}} R^2 - 6 \mu_m D_m \frac{C_m}{R^3} - \mu_m \frac{\lambda_{\eta m} - \lambda_m}{\alpha_m H_m} E_m \left[\frac{1}{2} J_0 (k, R) + J_2 (k, R) + \frac{1}{2} J_4 (k, R)\right]$$

$$- \mu_m \frac{\lambda_{\eta m} - \lambda_m}{\alpha_m H_m} F_m \left[\frac{1}{2} Y_0 (k, R) + Y_2 (k, R) + \frac{1}{2} Y_4 (k, R)\right],$$  \hfill (30)

$$\Sigma_{p\mu m} = -2 \mu_m A_m - \frac{3 \mu_m B_m R^2}{1 - \nu_{\mu m}} - \mu_m \frac{C_m}{1 - \nu_{\mu m}} R^2 - 6 \mu_m \frac{D_m}{R^4} + \mu_m \frac{\lambda_{\eta m} - \lambda_m}{\alpha_m H_m} E_m \left[\frac{1}{2} J_0 (k, R) - \frac{1}{2} J_4 (k, R)\right]$$

$$+ \mu_m \frac{\lambda_{\eta m} - \lambda_m}{\alpha_m H_m} F_m \left[\frac{1}{2} Y_0 (k, R) - \frac{1}{2} Y_4 (k, R)\right],$$  \hfill (31)

$$P_{p\mu m} = \frac{\lambda_{\eta m} + \lambda_m}{\alpha_m} \left[-\left(6 - \frac{3}{2 \nu_{\mu m}}\right) B_m R^3 + \frac{1}{1 - \nu_{\mu m}} \frac{C_m}{R^3}\right] + \frac{\lambda_{\eta m} + \lambda_m}{\alpha_m H_m} \frac{1}{2k_m} E_m J_2 (k, R) + F_m Y_2 (k, R)],$$  \hfill (32)

$$W_{\mu m} = \frac{\nu_{\mu m}}{\nu_{\alpha m}} \frac{\nu_{\alpha m}}{\alpha_m} \left[1 - 2 \nu_{\mu m}\right] B_m R^3 + \frac{\nu_{\mu m}}{\nu_{\alpha m}} \frac{1}{1 - \nu_{\alpha m}} \frac{C_m}{R^3} \left[E_m (J_1 (k, R) - J_3 (k, R)) + F_m (Y_1 (k, R) - Y_3 (k, R))\right].$$  \hfill (33)

in the matrix phase, and

$$U_{\eta c} = A_c R + \frac{D_c}{R^3},$$  \hfill (34)

$$U_{p\mu c} = -A_c R + \frac{D_c}{R^3},$$  \hfill (35)

$$\Sigma_{\eta c} = 2 \mu_c A_c - 6 \mu_c \frac{D_c}{R^3},$$  \hfill (36)
\[\Sigma_{\psi} = -2\mu_i A_i - 6\mu_e \frac{D_e}{R^4}, \quad (37)\]
in the effective medium, where \(k_i\) and \(k_m\) are wave numbers in the inclusion and matrix, respectively. \(\nu_{ui}\) and \(\nu_{um}\) are the undrained Poisson’s ratios of the inclusion and the matrix, respectively. \(\mu_i\) and \(\mu_m\) are the shear moduli of the inclusion and the matrix, respectively. \(\mu_e\) is the effective shear modulus to be determined. \(H_{i0}(=\lambda_i + 2\mu_i)\) and \(H_{m0}(=\lambda_m + 2\mu_m)\) are drained P-wave moduli of the inclusion and the matrix, respectively. In these equations, the unknown coefficients \(A_i, B_i, E_i, A_m, B_m, C_m, D_m, E_m, F_m, A_e\) and \(D_e\) can be determined from the boundary conditions at inclusion-matrix and matrix-effective interfaces.

We use the standard interface conditions given by Deresiewicz & Skalak (1963) to describe the physical continuity at inclusion-matrix interface. At the inclusion-matrix interface \((R = a)\), six independent continuity conditions involve the continuity of the displacements \(u_R\) and \(u_\theta\), the stresses \(\sigma_{RR}, \sigma_{R\theta}\), the pore pressure \(p_f\) and the radial filtration displacement \(w_R\). These six continuity conditions lead to following equations sequentially:

\[U_{Ri} (a) = U_{Rm}(a), \quad (38)\]
\[U_{ji} (a) = U_{jm}(a), \quad (39)\]
\[\Sigma_{Ri} (a) = \Sigma_{Rm}(a), \quad (40)\]
\[\Sigma_{ji} (a) = \Sigma_{jm}(a), \quad (41)\]
\[P_{fi} (a) = P_{fm}(a). \quad (42)\]
\[W_{Ri} (a) = W_{Rm}(a). \quad (43)\]

Furthermore, at the interface between the matrix and the effective medium \((R = b)\), four independent boundary conditions involve the continuity of the displacements \(u_R\) and \(u_\theta\) and the stresses \(\sigma_{RR}\) and \(\sigma_{R\theta}\). The continuity of \(u_R, u_\theta, \sigma_{RR}\) and \(\sigma_{R\theta}\) at \(R = b\) gives the following equations sequentially:

\[U_{Rm} (b) = U_{Rb}(b), \quad (44)\]
\[U_{jm} (b) = U_{jb}(b), \quad (45)\]
\[\Sigma_{Rm} (b) = \Sigma_{Rb}(b), \quad (46)\]
\[\Sigma_{jm} (b) = \Sigma_{jb}(b). \quad (47)\]

Now, we have ten boundary conditions owing to the continuity at the two interfaces. To determine the eleven unknown coefficients, one closure boundary condition at \(R = b\) is required. In analogy to the problem of spherical inclusion (Song et al. 2016), we discuss two possible situations for the required boundary condition.

**Situation 1.** Open pore (self-consistent or drained) boundary condition: \(p_f = 0\) at \(R = b\).

For the present problem of poroelastic media with pure shear stress, the averaged pore fluid pressure is zero. The open pore (or self-consistent) condition means that the pore fluid pressure along the effective medium surface is identical to the averaged pore fluid pressure. This situation leads to a drained form

\[P_{fm} (b) = 0. \quad (48)\]

**Situation 2.** Sealed pore (pore-solid or undrained) boundary condition: \(w_R = 0\) at \(R = b\).

Recall that we assume the outer effective medium to be an equivalent viscoelastic or elastic solid. The sealed pore condition means that there is no fluid mass flowing across the interface between the poroelastic matrix and the effective solid. This situation leads to an undrained form

\[W_{Rb} (b) = 0. \quad (48)\]

We note that the appropriate fifth boundary condition at \(R = b\) is an open issue. In numerical homogenization technique, Jänicke et al. (2015) used periodic boundary conditions, which is based on Hill-Mandel criterion for their finite element modeling.

By using eqs (22)–(37), eqs (38)–(48) can be rewritten in the matrix form as

\[b = 0, \quad (49)\]

where \(b\) is a \(11 \times 11\) matrix with complex elements, \(y\) is a column vector of eleven unknown coefficients \((A_i, B_i, E_i, A_m, B_m, C_m, D_m, E_m, F_m, A_e\) and \(D_e\)). The objective is to find the effective shear modulus \(\mu_e\), such that the determinant of matrix \(b\) is identical to zero.
Dynamic transverse shear modulus $\mu_e$. It is worth mentioning that the choice of $b$ is not unique owing to the properties of elementary transformation of matrix. Alternative and dimensionless choices of the elements of $b$ are shown in Appendix B. Based on the elements shown in Appendix B, the equation for determining the effective shear modulus $\mu_e$ is

$$A_1 \left( \frac{\mu_e}{\mu_m} \right)^2 + A_2 \frac{\mu_e}{\mu_m} + A_3 = 0 \quad (50)$$

with

$$A_1 = 3b_{10,10,9,11}$$
$$A_2 = b_{7,10,10,11} + 2b_{8,10,10,11} - b_{9,10,8,11} - 2b_{7,10,9,11}$$
$$A_3 = -b_{8,10,7,11} \quad (51)$$

where $b_{r,s,t,u}$ ($r, s = 7, 8, 9, 10$) is the determinant of generated matrix of $b$ by removing elements in $r$th and $s$th rows and in 10th and 11th columns.

Relation (50) is the final form sought. Using the quadratic formula, one can determine the exact solution for effective shear modulus of the cylindrical composite.

4 EXAMPLES

Dispersion and attenuation of the derived transverse shear modulus $\mu_e$ are calculated in this section. The $S$-wave quality factor $Q_s$ is expressed as

$$Q_s = -\frac{\text{Re}(\mu_e)}{\text{Im}(\mu_e)} \quad (52)$$

where Re and Im denote the real and imaginary parts. The inverse quality factor $Q_s^{-1}$, which represents the fraction of wave energy lost to heat in each wave period, describes the intrinsic attenuation in $S$ waves.

We start this section with two comparison examples with 2-D (plane strain) finite-element modeling results, which are previously solved by Quintal et al. (2012). The geometric model in their numerical simulations consists of a single square unit cell or periodic distribution of unit cells. Their single unit cell (shown in Fig. 3) has a circular inclusion with much lower porosity and permeability than the continuous matrix material, which contains 80 per cent of the total pore space of the unit cell. The inclusion is always fully saturated with water, whereas the matrix is fully saturated with gas, oil or water. The unit cell of this model is a 0.4 m side square, and the radius of the circular inclusion is 0.16 m. The inclusion occupies 50 per cent volume of the unit cell. The physical parameters used in their simulation are shown in Tables 1 and 2, which show the physical properties of the pore fluids and solid frames, respectively. Table 2 shows that the inclusion with lower porosity and permeability than the matrix is also stiffer than the matrix.

To compare with their simulation results, we let $\alpha = 0.16$ m and $\beta = 0.5$ in our three-phase model and use the same physical parameters for calculation. Figs 4–6 compare the results of finite-element modeling and the results of the three-phase model. The legend terms in Figs 4 and 5 refer to the three saturation scenarios: 80 per cent gas, 20 per cent water (G); 100 per cent water (W) and 80 per cent oil, 20 per cent water (O).

Figure 3. A schematic picture of Quintal et al.’s (2012) single unit cell.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Water</th>
<th>Oil</th>
<th>Gas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_f$ (GPa)</td>
<td>2.4</td>
<td>1.4</td>
<td>0.04</td>
</tr>
<tr>
<td>$\eta$ (Pa s)</td>
<td>0.001</td>
<td>0.02</td>
<td>$2 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Table 2. Physical properties of the solid frames for the inclusion and matrix (Quintal et al. 2012). Utilization for calculation in Figs 4–6.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Inclusion</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$ (GPa)</td>
<td>36</td>
<td>4</td>
</tr>
<tr>
<td>$\mu$ (GPa)</td>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>$K_s$ (GPa)</td>
<td>40</td>
<td>48</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.06</td>
<td>0.26</td>
</tr>
<tr>
<td>$\kappa$ (mD)</td>
<td>40</td>
<td>1000</td>
</tr>
</tbody>
</table>

Figure 4. Real part of the dynamic transverse shear modulus and $S$-wave quality factor. Part (a) of the figure shows the finite-element results of a model consisting of 25 square unit cells spatially arranged in a $5 \times 5$ array with undrained external boundary from Quintal et al. (2012). Part (b) of the figure shows the analytical results of the present three-phase model with open pore (or self-consistent) boundary. The fluid and frame properties are given in Tables 1 and 2. The legend terms refer to the three saturation scenarios: 80 per cent gas, 20 per cent water (G); 100 per cent water (W); and 80 per cent oil, 20 per cent water (O).

Fig. 4(a) shows the finite-element results (from Quintal et al. 2012) for a model consisting of 25 square unit cells spatially arranged in a $5 \times 5$ array with undrained external boundary, while Fig. 4(b) shows the analytical results of the present three-phase model with open pore boundary which is discussed in Section 3. Fig. 5(a) shows the finite-element results (from Quintal et al. 2012) for a model consisting of a single square unit cell with undrained external boundary, while Fig. 5(b) shows the analytical results of the present three-phase model with sealed pore boundary. The comparison in Figs 4 and 5 shows there is very good agreement for relaxation frequency between finite-element modeling and our three-phase model, although the magnitudes are obviously different. The relaxation frequency is the frequency at which the minimum value of the quality factor occurs. The three-phase model with self-consistent boundary is sufficient to model the relaxation frequency of a medium with periodically distributed circular inclusions embedded in a homogeneous matrix, while the three-phase model with sealed boundary is sufficient to model the relaxation frequency of undrained unit cell. Moreover, the relative positions of Re($\mu_e$) and $Q_s$ for different pore fluids are also very similar to those in the finite-element results. In Figs 4(b) and 5(b), we also observe that the real part of the shear modulus $\mu_e$ at the low-frequency limit is independent of the properties of the saturating fluid. This property comes from the fact that the pore fluids have no effect on the shear properties of the porous material at low frequencies. In contrast, the values of Re($\mu_e$), which increase with frequency, are different for different pore fluids at relative higher frequencies. The $S$-wave attenuation in the gas-saturated media is very low and much lower than in the fully water-saturated and partially oil-saturated media. Moreover, at frequencies equal to or lower than the relaxation frequencies in the oil-saturated media, the $S$-wave attenuation is about one order of magnitude higher in the partially oil-saturated media than in the fully water-saturated media. Therefore, the pore fluids could cause significant effects on the dispersion and attenuation in the combined model.

There are some differences between finite-element modeling and the analytical three-phase model. Fig. 4 shows that the three-phase model exhibits more attenuation and higher modulus values as well as larger dispersion than the periodically distributed unit cells, whereas Fig. 5 shows that the three-phase model exhibits lower attenuation and modulus values as well as smaller dispersion than the single unit cell.
Dynamic transverse shear modulus

Figure 5. Real part of the dynamic transverse shear modulus and S-wave quality factor. Part (a) of the figure shows the finite-element results of a model consisting of a single square unit cell with undrained external boundary from Quintal et al. (2012). Part (b) of the figure shows the analytical results of the present three-phase model with sealed pore boundary. The fluid and frame properties are given in Tables 1 and 2. The legend terms refer to the three saturation scenarios: 80 per cent gas, 20 per cent water (G); 100 per cent water (W); and 80 per cent oil, 20 per cent water (O).

To better compare the finite-element modeling and this study, we put the results of this study and finite-element modeling in the same diagram (Fig. 6). Figs 6(a) and (b), respectively plot \( \text{Re}(\mu_e) \) versus frequency and \( Q_s^{-1} \) versus frequency for 100 per cent water saturation. For frequencies lower than relaxation frequencies, they show that values of \( \text{Re}(\mu_e) \) and \( Q_s^{-1} \) for the three-phase model fall between results of single unit cells and results of periodically distributed unit cells. But, unlike the case of periodically distributed unit cells in which the S-wave dispersion and attenuation are very weak, the three-phase model causes more attenuation and modulus dispersion. We also observe that the sealed-pore situation predicts more attenuation than the open-pore situation at relatively lower frequencies (i.e., frequencies lower than relaxation frequencies).

Note that Figs 4–6 show the results for porous solids containing an inclusion having lower porosity and permeability than the matrix. In addition, both the solid frame and the pore fluid are different in the matrix and inclusion. It is unclear that whether the heterogeneity in the solid or the heterogeneity in the fluid dominates the attenuation. Next, we use the three-phase model to calculate the modulus dispersion and inverse quality factor for another two cases:

1. The matrix has lower porosity and permeability than the inclusion, but the pore fluid is the same in the matrix and inclusion (A double porosity model having stiffer matrix than the inclusion). This case has been widely used to describe the properties of heterogeneous fluid-filled rocks (Liu et al. 2009; Ba et al. 2011).
2. The pore fluids are different, but the matrix has the same porosity and permeability as the inclusion (A patchy saturation model).

Table 3 lists physical parameters of solid frames for the inclusion and matrix. The two porous media differ by porosity, permeability and elastic moduli. The drained stiffnesses of the high porosity medium are only 1 per cent of the low porosity medium. The matrix represents a stiff porous solid, while the inclusion represents a relatively softer porous solid. These parameters are used to compute the attenuation and dispersion in Figs 7–10.

Parts (a) and (b) of Fig. 7 shows the dispersion curves of the real part of the effective transverse shear modulus and inverse quality factor for a double-porosity model with a fixed inclusion concentration \( \beta = 0.1 \) and outer radii of \( b = 0.01, 0.1 \) and 1 m, respectively. In this double-porosity model, both the inclusion and the matrix are water saturated, but the frame stiffness of inclusion is only 1 per cent of the matrix (Table 3). The results of open pore situation (black lines) and sealed pore situation (blue lines) are compared in Fig. 7. It is shown that the real part of the effective shear modulus and the inverse quality factor curves for open pore situation are very similar to the sealed pore situation. It is shown that the real part of the effective shear modulus at the high frequency limit is higher than the modulus at the low frequency limit and the modulus increases with frequency. Moreover, the frequency dependences of the modulus and the attenuation show that the relaxation frequency moves towards higher frequencies when the outer radius becomes smaller. This indicates that the relaxation frequency is size-dependent. More precisely, the relaxation frequency moves towards higher frequencies by two orders of magnitude when the outer radius becomes smaller by one order of magnitude. In the present three-phase model, the intrinsic S-wave attenuation is caused by...
Figure 6. Comparison of the real part of the dynamic transverse shear modulus and attenuation between this study and finite-element results of Quintal et al. (2012) for 100 per cent water saturation. (a) Real part of the shear modulus. (b) Inverse quality factor of the shear modulus. The fluid and frame properties are given in Tables 1 and 2.

Table 3. Physical properties of the solid frames for the inclusion and matrix (Song et al. 2016). Utilization for calculation in Figs 7–10.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Inclusion</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$ (GPa)</td>
<td>0.13</td>
<td>13</td>
</tr>
<tr>
<td>$\mu$ (GPa)</td>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>$K_s$ (GPa)</td>
<td>38</td>
<td>38</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.3</td>
<td>0.15</td>
</tr>
<tr>
<td>$\kappa$ (mD)</td>
<td>13 000</td>
<td>130</td>
</tr>
</tbody>
</table>

slow (diffusive) $P$ wave. This characteristic is the same as $P$ wave (Pride et al. 2004) and $S$-wave attenuation caused by spherical inclusion (Song et al. 2016).

Next, we look at the differences between the two distinct pore situations. Fig. 7 shows that the differences of the modulus dispersion and attenuation curves only occur at relatively low frequencies. The sealed pore situation has higher modulus and more attenuation than the open pore situation at relative low frequencies. This is because the sealed pore situation inhibits fluid pressure relaxation more than the open pore situation. This characteristic is also seen in Figs 4(b) and 5(b). However, the maximum attenuation in Fig. 7(b) which represents a soft inclusion imbedded in a relatively stiff matrix is much smaller than those in Figs 4(b) and 5(b) which represents a relatively stiff inclusion imbedded in a soft matrix.
Figure 7. Real part of the dimensionless effective transverse shear modulus (a) and S-wave inverse quality factor (b) of a double-porosity model with a fixed inclusion concentration $\beta = 0.1$ but various radius $b = 0.01, 0.1$ and 1 m, respectively. Black lines and blue lines are results for the open pore situation and the sealed pore situation, respectively. The effective shear moduli in red and green lines in (a) are calculated by means of Christensen & Lo (1979) using drained and undrained parameters (Poisson’s ratios), respectively. Both the matrix and inclusion are water saturated. Physical parameters are shown in Table 3.

To get more physical insights into the lower-frequency and higher-frequency moduli, two curves of shear modulus (green and red lines) calculated by means of classic three-phase elasticity model (Christensen & Lo 1979) are also plotted in Fig. 7(a). The effective shear modulus of Christensen & Lo’s (1979) elasticity model depends on the shear moduli and Poisson’s ratios of matrix and inclusion as well as the inclusion concentration. Detailed formula for the effective shear modulus in the classic elasticity model are given in Appendix C. In our calculation procedure, shear moduli of the matrix and inclusion and the inclusion concentration used in the elasticity model are all the same as those in the poroelastic model, but the Poisson’s ratios of the matrix and inclusion used to calculate the effective shear moduli in the red and green lines are equal to the drained and undrained values, respectively. For both the open pore and the sealed pore situations, Fig. 7(a) shows that the drained effective shear modulus (red line) and undrained effective shear modulus (green line) are exactly equal to the low-frequency and high-frequency shear moduli of the poroelastic composite material, respectively. This implies that with the increase of frequency, the double-porosity composite transitions from a composite having drained inclusions and drained matrix with no pore pressure difference to another having undrained inclusions and undrained matrix with no fluid communication. This frequency-dependent characteristic is caused by the fluid pressure relaxation.

Now, we consider the magnitude of attenuation in the double-porosity model. Fig. 8 plots the curves of S-wave inverse quality factor versus frequency for a double-porosity model with a fixed radius $b = 0.5$ m but different inclusion concentrations. In this figure, physical properties of the matrix and the inclusion are shown in Table 3. To quantify the attenuation for a wide range of inclusion concentration, we choose ten values of $\beta = 0.01, 0.02, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ and 0.7. Figs 8(a) and (b) give the results for the open pore and sealed pore situations, respectively. The asymptotic characteristics for the two boundary situations are very similar to each other. For example, at low frequencies inverse quality factor scales with the first power of frequency $\omega$, while at high frequencies the attenuation is proportional...
Figure 8. Log–log plot of S-wave inverse quality factor $Q_s^{-1}$ versus angular frequency $\omega$ for a double-porosity model with fixed outer radius $b = 0.5$ m but various inclusion concentrations of $\beta = 0.01, 0.02, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ and 0.7, respectively. Part (a) of the figure shows $Q_s^{-1}$ for the open situation. Part (b) of the figure shows $Q_s^{-1}$ for the sealed pore situation. Both the matrix and the inclusion are water-saturated. Physical parameters are shown in Table 3.

to $\omega^{-1/2}$. Moreover, for small inclusion concentration, the magnitude of inverse quality factor increases with inclusion concentration. Note, from Fig. 8, that the attenuation maximum occurs in a curve in which the inclusion concentration equals to 0.5. The inverse quality factor maximum is nearly 0.03 for open pore situation, while it is nearly 0.02 for sealed pore situation. In contrast, Fig. 6 shows that significant attenuation can occur much more easily for the case of a stiff inclusion imbedded in a soft matrix.

Here, it is of great interest to compare the S-wave attenuation in this cylindrical model caused by mesoscopic WIFF with S-wave attenuation by Biot global-flow mechanism (Biot 1956a,b) and S-wave attenuation in previous spherical model (Song et al. 2016). The attenuation in Biot mechanism is caused by the viscous friction of fluid motion relative to the solid. The attenuation in the spherical model and the present cylindrical model is caused by mesoscopic WIFF. Fig. 9 plots the S-wave inverse quality factors for the three models. The Biot attenuation is calculated using Johnson et al.‘s (1987) theory of dynamic permeability. Physical properties used for calculation are shown in Table 3, the inclusion concentration and outer radius are chosen as $\beta = 0.5$ and $b = 0.5$ m. Fig. 9(a) compares the results for the open pore situation, while Fig. 9(b) compares the results for the sealed pore situation. The black and red lines in Fig. 9 represents the S-wave attenuation in the present cylinder model and in the sphere model, respective, and the blue line represents the Biot S-wave attenuation in the matrix material. For both of the two pore boundary situations, the comparison in Fig. 9 shows that the attenuation maximum of the cylinder and sphere model occurs at angular frequencies which approximately equal to $10^3$ rad s$^{-1}$, whereas the Biot S-wave attenuation maximum occurs at very high angular frequencies ($10^5$–$10^6$ rad s$^{-1}$). Therefore, the major S-wave attenuation caused by mesoscopic WIFF can occurs in seismic bands. Moreover, Fig. 9 shows that the S-wave attenuation in the cylinder model is apparently larger than the sphere model.

Last, we look at the patchy-saturation model. $k_m$ stands for the Biot-diffusion wavenumber of the matrix. Fig. 10 plots the S-wave inverse quality factor versus a dimensionless variable Re($k_m b$) for the sealed pore situation with an inclusion concentration $\beta = 0.15$ and outer radius $b = 0.4$ m. The result of a patchy-saturation model (dash line) is compared with the result of a double-porosity model (solid line). For the double-porosity model (a soft inclusion imbedded in stiff matrix), both the matrix and the inclusion are water saturated. Physical parameters are shown in Table 3. For the patchy-saturation model, the matrix is water-saturated and the inclusion is gas-saturated, but the solid frame has
Figure 9. Comparison of S-wave inverse quality factor between the present double-porosity cylinder model (black line), the double-porosity sphere model (red line) and the Biot global fluid mechanism (blue line). The outer radius and inclusion concentration are selected by $b = 0.5$ m and $\beta = 0.5$. Both the matrix and the inclusion are water saturated. Physical parameters are shown in Table 3. Part (a) of the picture shows the results of open pore situation. Part (b) of the picture shows the results of sealed pore situation.

Figure 10. Comparison of S-wave inverse quality factor between a double-porosity model (solid line) and a patchy-saturation model (dash line) with the same inclusion concentration $\beta = 0.15$. For the double-porosity model, both the matrix and the inclusion are water saturated. Physical parameters are shown in Table 3. For the patchy-saturation model, the matrix is water-saturated and the inclusion is gas-saturated, but the solid frame has the same properties as the double-porosity matrix.
the same properties as the double-porosity matrix. Fig. 10 shows that there is no attenuation in the patchy-saturation model. We also calculate the $S$-wave inverse quality factor in the patchy-saturation model for open pore situation, the effective shear modulus is also non-dissipative.

5 CONCLUSIONS

This paper investigates the effect of mesoscopic WIFF on effective shear properties in a heterogeneous fluid-filled porous material with cylindrical inclusions. An analytical solution is presented for the effective transverse shear modulus in a three-phase or generalized self-consistent model under the assumption that the diameter of a circular unit cell is smaller than $S$-wave wavelength but larger than the size of pores. The heterogeneity can be addressed in either the solid frame, the pore fluid or both.

For the double-porosity model, we have studied two cases of inclusion: (1) the solid frame of the inclusion is stiffer than the matrix frame and (2) the solid frame of the inclusion is softer than the matrix frame. The case of the relatively stiff inclusion imbedded in a soft matrix can cause significant $S$-wave attenuation at seismic bands. For the effective shear properties, the heterogeneous poroelastic material model transition with the increase of frequency from a composite having drained inclusions and drained matrix with no pore pressure gradient to the same medium having undrained inclusions and undrained matrix with no fluid communication.

For the patchy-saturation model, the heterogeneity in fluid does not cause any $S$-wave attenuation at all.

The solution is restricted to a plane-strain problem. Although actual reservoirs are not so ideal, the analytical solution is helpful for validating numerical methods. Because governing equations of thermoelasticity are mathematically equivalent to those of poroelasticity (Biot 1956c and Rice & Cleary 1976; see also Berryman & Milton 1991), this study and the companion paper (Song et al. 2016) can be modified for corresponding problems of thermoelasticity with dispersion between isothermal and adiabatic states.

ACKNOWLEDGEMENTS

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APPENDIX A: DERIVATIONS OF THE CHANGE IN FLUID VOLUME CONTENT AND DISPLACEMENTS

Suppose an \( e^{-\omega t} \) dependence. In a polar coordinate system the fluid diffusion eq. (5) can be expressed as

\[
\frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial U}{\partial R} \right) - \frac{\partial^2 U}{\partial Z^2} = \frac{\omega U}{c^2}.
\]  

(A1)

Inserting eq. (9) \( \zeta = Z(R) \cos 2\theta \) into eq. (A1) leads to a Bessel equation of the second order

\[
\left( \frac{d^2}{d\xi^2} + \frac{1}{\xi} \frac{d}{d\xi} \left( \frac{1 - 4}{\xi^2} \right) \right) Z = 0,
\]  

(A2)

where \( \xi = kR, k = \sqrt{\omega/c} \). So, the solutions for \( Z \) and \( \xi \) are

\[
Z = E J_2(kR) + FY_2(kR),
\]  

(A3)

\[
\zeta = [E J_2(kR) + FY_2(kR)] \cos 2\theta,
\]  

(A4)

where \( J_2 \) and \( Y_2 \) are two-order Bessel functions of two kinds, \( E \) and \( F \) are constants to be determined.

Next, substituting displacements given by eqs (7) and (8) into equilibrium eq. (4) gives two coupled linear, non-homogeneous ordinary differential equations

\[
\frac{d^2 U_R}{dR^2} + \frac{1}{R} \frac{d U_R}{dR} - \frac{5 U_R}{R^2} = \frac{4 U_R}{R^2} - 2 U_\phi + \frac{2dU_\phi}{dR} - \frac{2U_\phi}{R^2} = \frac{\lambda u - \lambda dZ}{\alpha \mu} \frac{dR}{dR},
\]  

(A5)

\[
\frac{d^2 U_\phi}{dR^2} + \frac{1}{R} \frac{d U_\phi}{dR} - \frac{5 U_\phi}{R^2} = \frac{4 U_\phi}{R^2} - \frac{2}{1 - 2v_u} \left( \frac{1}{R} \frac{d U_R}{dR} + \frac{U_R}{R^2} + \frac{2 U_\phi}{R^2} \right) = -\frac{2 \lambda u - \lambda dZ}{\alpha \mu},
\]  

(A6)

where \( v_u \) is the undrained Poisson’s ratio. The coupled eqs (A5) and (A6) can be noted as

\[
\Gamma_1(U) + \Gamma_2(U) = \chi,
\]  

(A7)

where

\[
\Gamma_1 = \left( \frac{d^2 U_R}{dR^2} + \frac{1}{R} \frac{d U_R}{dR} - \frac{5 U_R}{R^2} \right),
\]  

(A8)

\[
\Gamma_2 = \frac{1}{1 - 2v_u} \left( \frac{d^2 U_R}{dR^2} + \frac{1}{R} \frac{d U_R}{dR} - \frac{U_R}{R^2} + \frac{2dU_\phi}{dR} - \frac{2U_\phi}{R^2} \right),
\]  

(A9)

\[
\chi = \frac{\lambda u - \lambda dZ}{\alpha \mu} \left( \frac{dR}{dR} \right).
\]  

(A10)

The general solutions for displacements in the homogeneous equation \( \Gamma_1 + \Gamma_2 = 0 \) are shown in Christensen & Lo (1979). We write them as

\[
U_R^0 = AR + BR^2 + \frac{C}{R} + \frac{D}{R^3},
\]  

(A11)
Eliminating the term \( U_0^i \) in (A13) gives

\[
\frac{\lambda_u - \lambda}{\alpha \mu} \left[ \frac{d^2 Z}{dR^2} + \frac{5}{R} \frac{d^2 Z}{dR^2} + \frac{1}{R^2} \frac{d Z}{dR} - \frac{8 Z}{R^3} \right] = \frac{d}{dR} \left[ \frac{1}{R} \frac{d}{dR} \left( \frac{1}{R} \frac{d}{dR} \left( R^3 U_0^i \right) \right) \right].
\]

(A14)

By using eq. (A3), we have the solutions for \( U^i \):

\[
\begin{align*}
U_0^i &= \frac{\lambda_u - \lambda}{\alpha \mu} \frac{1}{2k} \left( E [J_i (kR) - J_i (kR)] + F [Y_i (kR) - Y_i (kR)] \right) , \\
U_0^j &= \frac{\lambda_u - \lambda}{\alpha \mu} \frac{1}{4k} \left( E [J_i (kR) - J_i (kR)] + F [Y_i (kR) - Y_i (kR)] \right).
\end{align*}
\]

(A15)

Next, consider the equation \( \Gamma_1(U^i) + \Gamma_2(U^i) = -\Gamma_3(U^i) \), where

\[
\Gamma_2(U^i) = \frac{k \lambda_u - \lambda}{\alpha \mu} \left( E [J_i (kR) - J_i (kR)] + F [Y_i (kR) - Y_i (kR)] \right) .
\]

(A16)

Moreover, the special solutions for \( U^i \) in eq. (A7) are the sum of \( U^i \) and \( U^j \):

\[
\begin{align*}
U_0^i &= \frac{1}{4(1 - \nu_u)} \frac{\lambda_u - \lambda}{\alpha \mu} \frac{k}{\mu} \left( E [J_i (kR) - J_i (kR)] + F [Y_i (kR) - Y_i (kR)] \right), \\
U_0^j &= \frac{1}{4(1 - \nu_u)} \frac{\lambda_u - \lambda}{\alpha \mu} \frac{k}{\mu} \left( E [J_i (kR) + J_i (kR)] + F [Y_i (kR) + Y_i (kR)] \right) .
\end{align*}
\]

(A17)

(A18)

where the relation \( \frac{1}{1 + 2 \nu_u} = \frac{1}{R} \) is used in eq. (11).

Last, the general solutions for displacements in the non-homogeneous eq. (A7) are

\[
U = U^0 + U^i .
\]

(A19)

### APPENDIX B: ELEMENTS IN 11 \times 11 MATRIX \( b \)

The choice of elements in matrix \( b \) is not unique due to the properties of elementary transformation applied in eqs (38)–(48). A group of non-dimensional elements are collected here.

For the situation of open pore boundary \( (p_f = 0) \) at \( R = b \)

\[
\begin{align*}
b_{11} &= 1 , b_{12} = v_{uu} , b_{13} = \frac{1}{(a_k)^2} \cdot [J_0 (k_R) - J_1 (k_R)] , b_{14} = -1 , b_{15} = -v_{uu} , b_{16} = v_{uu} - 1 , b_{17} = -1 , \\
b_{18} &= (a_k)^2 \cdot [J_1 (k_R) - J_2 (k_R)] , b_{19} = (a_k)^3 \cdot [Y_1 (k_R) - Y_2 (k_R)] , b_{21} = 1 , b_{22} = 1.5 - v_u , \\
b_{23} &= (a_k)^3 \cdot [J_1 (k_R) - J_2 (k_R)] , b_{24} = -1 , b_{25} = v_{uu} - 1.5 , b_{26} = v_{uu} - 0.5 , b_{27} = 1 , b_{28} = (a_k)^3 \cdot [J_1 (k_R) + J_2 (k_R)] , \\
b_{29} &= (a_k)^3 \cdot [Y_1 (k_R) + Y_2 (k_R)] , b_{31} = 1 , b_{33} = 0 , b_{34} = \frac{1}{2} J_0 (k_R) + J_2 (k_R) , b_{35} = \frac{1}{2} J_2 (k_R) , b_{36} = \frac{1}{2} J_0 (k_R) , b_{37} = -\frac{1}{2} J_0 (k_R) - J_2 (k_R) , \\
b_{38} &= \frac{1}{2} [J_0 (k_R) - J_2 (k_R)] , b_{41} = -1 , b_{42} = 0.5 \cdot J_0 (k_R) - J_2 (k_R) , b_{44} = \mu_u / \mu_i , b_{45} = 1.5 \mu_u / \mu_i , b_{46} = 0.5 \mu_u / \mu_i , b_{47} = 3 \mu_u / \mu_i , b_{48} = 0.5 \cdot J_0 (k_R) + J_2 (k_R) \mu_u / \mu_i , b_{51} = 3 \mu_u / \mu_i , b_{52} = 3 (1 - 2 v_{uu}) , b_{53} = 2 \frac{H_d}{k_{uu}} - J_2 (k_R) , \\
b_{54} &= 0.5 \cdot J_0 (k_R) - J_2 (k_R) \mu_u / \mu_i , b_{55} = 0.5 \cdot J_2 (k_R) + J_0 (k_R) \mu_u / \mu_i .
\end{align*}
\]
The solution for the effective shear modulus \( \mu \) three-phase elasticity model and the remaining elements are identical to zero, where \( \beta = (a/b)^2 \).

For the situation of sealed pore boundary \((w_s = 0)\) at \( R = b \), only four elements are different from these in the situation of open pore boundary. They are

\[
b_{11.5} = 6 (1 - 2v_m) \beta^{-1}, \quad b_{11.6} = -2 (1 - 2v_m) \beta, \quad b_{11.8} = k_m b \frac{H_{im}}{\lambda_m - \lambda_m} [J_1 (k_m b) - J_3 (k_m b)],
\]

\[
b_{11.9} = k_m b \frac{H_{im}}{\lambda_m - \lambda_m} [Y_1 (k_m b) - Y_5 (k_m b)].
\]

**APPENDIX C: EFFECTIVE TRANSVERSE SHEAR MODULUS IN THE CYLINDRICAL THREE-PHASE ELASTICITY MODEL**

The solution for the effective shear modulus \( \mu_e \) in an elastic three-phase composite cylinder is given by Christensen & Lo (1979). The solution can be written formally as

\[
\tilde{A}_1 \left( \frac{\mu_e}{\mu_0} \right)^2 + \tilde{A}_2 \frac{\mu_e}{\mu_0} + \tilde{A}_3 = 0,
\]

where the constants \( \tilde{A}_1, \tilde{A}_2, \) and \( \tilde{A}_3 \) are given by

\[
\tilde{A}_1 = 3\beta(1 - \beta)^2 \left( \frac{\mu_1}{\mu_0} - 1 \right) \left( \frac{\mu_1}{\mu_0} + \eta_1 \right) + \left[ \frac{\mu_1}{\mu_0} + \eta_1 \eta_0 + \eta_1 \eta_0 \right] \right] \left[ \beta \eta_0 \left( \frac{\mu_1}{\mu_0} - 1 \right) - \left( \frac{\mu_1}{\mu_0} + \eta_1 \right) \right],
\]

\[
\tilde{A}_2 = -6\beta(1 - \beta)^2 \left( \frac{\mu_1}{\mu_0} - 1 \right) \left( \frac{\mu_1}{\mu_0} + \eta_1 \right) + \left[ 1 + \beta \left( \frac{\mu_1}{\mu_0} - 1 \right) + \frac{\mu_1}{\mu_0} \eta_0 \right] \right] \left[ (\eta_0 - 1) \left( \frac{\mu_1}{\mu_0} + \eta_1 \right) - 2\beta \left( \frac{\mu_1}{\mu_0} \eta_0 - \eta_1 \right) \right]
\]

\[
+ \beta \eta_0 \left[ \frac{\mu_1}{\mu_0} + \eta_1 + \beta \left( \frac{\mu_1}{\mu_0} \eta_0 - \eta_1 \right) \right],
\]

\[
\tilde{A}_3 = 3\beta(1 - \beta)^2 \left( \frac{\mu_1}{\mu_0} - 1 \right) \left( \frac{\mu_1}{\mu_0} + \eta_1 \right) + \left[ \frac{\mu_1}{\mu_0} + \beta \left( \frac{\mu_1}{\mu_0} - 1 \right) + 1 \right] \left[ \frac{\mu_1}{\mu_0} + \eta_1 + \beta \left( \frac{\mu_1}{\mu_0} \eta_0 - \eta_1 \right) \right].
\]

with

\[
\eta_1 = 3 - 4v_1, \quad (C5)
\]

\[
\eta_0 = 3 - 4v_0, \quad (C6)
\]

and where \( v_1 \) and \( v_0 \) are the Poisson’s ratios of the inclusion and matrix, respectively. \( \mu_1 \) and \( \mu_0 \) are the shear moduli of the inclusion and matrix, respectively. \( \beta \) is the volume fraction of the inclusion, that is, \( \beta = (a/b)^2 \).
When $\mu_1 = \mu_0$, the effective shear modulus $\mu_e$ satisfies the following equation:

$$B_1 \left( \frac{\mu_e}{\mu_0} \right)^2 + B_2 \frac{\mu_e}{\mu_0} + B_3 = 0,$$

(C7)

where

$$B_1 = \beta^3 (\eta_0 - \eta_1) - (1 + \eta_1) \eta_0,$$

(C8)

$$B_2 = (\eta_0 - 1)(1 + \eta_1) - 2\beta^3(\eta_0 - \eta_1),$$

(C9)

$$B_3 = 1 + \eta_1 + \beta^3(\eta_0 - \eta_1).$$

(C10)

The solution for eq. (C7) is

$$\left( \frac{\mu_e}{\mu_0} \right)_1 = 1,$$

(C11)

or

$$\left( \frac{\mu_e}{\mu_0} \right)_2 = \frac{1 + \eta_1 + \beta^3(\eta_0 - \eta_1)}{- (1 + \eta_1) \eta_0 + \beta^3(\eta_0 - \eta_1)}.$$  

(C12)

But the only suitable solution for $\mu_e$ is

$$\mu_e = \mu_0,$$

(C13)

because $\left( \frac{\mu_e}{\mu_0} \right)_2 < 0$ if $\beta = 0$ or if $\nu_0 = \nu_0$. 

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