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ON ENERGY RADIATION FROM SEISMIC SOURCES

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ABSTRACT

This note clarifies the relationships among various expressions for the energy radiated by elastodynamic seismic sources. The radiated energy can be expressed in terms of the far-field particle velocities provided that the stress-particle velocity relationship asymptotically approaches the plane wave relationship with increasing distance from the source. This condition is satisfied for all sources that can be characterized by a moment density tensor. For the case in which the source can be characterized as a point, i.e., the wavelengths of all emitted radiation are much greater than source dimensions, the radiated energy is expressed in terms of the moment tensor. The relation between these far-field representations and Kostrov's representation of radiated energy in terms of fault surface traction and particle velocity is established. Kostrov's representation is arranged in various forms to reveal the source of radiated energy as the deviations of the fault surface tractions and particle velocities from the values which would occur during quasi-static fault motion between the same end states. Moreover, the excess of the static strain energy change over the work done by the fault surface tractions, called W_0 by Kanamori, is shown to be a good approximation to the radiated energy when fault propagation speed is near the Rayleigh wave velocity and the time rate of change of fault surface traction is small.

INTRODUCTION

The determination of the energy released by an earthquake is one of the fundamental problems of seismology. Typically, empirical estimates of energy changes associated with earthquakes are obtained in two ways (Bath, 1966): from the transient dynamic displacements inferred from examination of seismograms or from measurements of the static deformation in the epicentral region. The first method yields an estimate of the seismic wave energy radiated to the far field whereas the second yields an estimate of the energy which has gone into overall deformation of the region around the source. Although the energy determined by the second method is frequently used as an approximation to that determined by the first, these energies are not, in general, equal and the relationship between them is often unclear.

The purpose of this note is to present various expressions for the radiated energy in order to illuminate its origin in seismic faulting and to clarify the relationship of radiated energy to other energies involved in earthquake faulting. The treatment is, however, limited to infinite linear elastic solids which are isotropic and homogeneous. Although such conditions are, of course, not satisfied in the earth, this idealization has nevertheless proven to be useful as an approximation and in providing guidance in more realistic problems.

Definition of Radiated Energy

The radiated energy is defined here as

$$E_R = \int_{-\infty}^{\infty} \int_S [-\sigma_{ij}\gamma_j \dot{u}_i] \, dS \, dt \tag{1}$$

where u_i is the change in displacement and σ_{ij} is the change in stress associated with u_i. The remaining symbols in (1) are defined as follows: t is time; the superposed dot denotes time differentiation; S is a spherical surface of radius r centered at the source with $r \gg l$, where l is a characteristic source dimension, and γ_i is the unit normal to S directed outward from the source. The integrand of (1) is simply the rate of work by the material inside S on the material outside S and this agrees with the fundamental definition of energy flow. If the stress change σ_{ij} and the particle velocity $\dot{u}_i$ are proportional to r^{-1} as r goes to infinity in any fixed direction, then E_R has a finite nonzero value in this limit.

An expression for radiated energy which is frequently used in seismology (e.g., Haskell, 1964) is

$$E_R = \int_{-\infty}^{\infty} \int_S \rho[c_d(\dot{u}_i\gamma_i)^2 + c_s(\dot{u}_i m_i)^2] \, dS \, dt \tag{2}$$

where ρ is the density, c_d and c_s are the dilatational and shear wave speeds, respectively, and m_i is the unit tangent vector to S in the direction of the shear traction. The definition (2) follows from (1) if

$$\dot{u}_i = 0(r^{-1}) \tag{3}$$

and

$$\sigma_{ij}\gamma_j + \rho\dot{u}_j\gamma_j\gamma_i c_d + \rho\dot{u}_j m_j m_i c_s = 0(r^{-2}) \tag{4}$$

as $r \to \infty$ in any fixed direction: because the area of S is proportional to r^2, terms which decay faster than r^{-2} do not contribute to (1) in the limit of $r \to \infty$. The condition (4) implies that far from the source, waves become asymptotic to plane waves in the sense that the traction and particle velocity are related as they are for plane waves or, in other words, that near the wave front the curvature of the wave front is negligible. The equivalence of (1) and (2) has been discussed for the special cases of harmonic plane waves (Bullen, 1963; Sagisaka, 1954) and spherical waves (Yoshiyama, 1963) and it has been demonstrated by direct calculation for a double-couple source (e.g., Dahlen, 1974). However, in the next section, it will be shown that (4) is satisfied, and hence (1) and (2) are equivalent, for any source which can be characterized by a moment density tensor.

Far-Field Representation

The usual approach of source theory (e.g., Archambeau, 1968; Backus and Mulcahy, 1976) is to model inelastic processes due to faulting, explosions, etc., as causing a region of an elastic body (the source) to undergo a stress-free change of size and shape without altering the elastic properties of the region. If this change of size and shape is expressed as a strain $\epsilon^T_{ij}(\underset{\sim}{x}, t)$ then the seismic moment density tensor is defined by (e.g., Kostrov, 1970; Aki and Richards, 1980, p. 59)

$$m_{ij}(\underset{\sim}{x}, t) = C_{ijkl}\,\epsilon^T_{kl}(\underset{\sim}{x}, t) \tag{5}$$

where C_{ijkl} is the tensor of elastic moduli. For an isotropic elastic solid the modulus tensor is given by

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \Lambda\delta_{ij}\delta_{kl} \tag{6}$$

where μ is the shear modulus and Λ is the Lame constant.

The displacements due to a given $m_{ij}(\underset{\sim}{x}, t)$ are then calculated as those due to an effective body force

$$f_i^{eff} = -\partial m_{ji}/\partial x_j. \tag{7}$$

The displacement field can be written as the sum

$$u_i(\underset{\sim}{x}, t) = u_i^d(\underset{\sim}{x}, t) + u_i^s(\underset{\sim}{x}, t) \tag{8}$$

where $u_i^d(\underset{\sim}{x}, t)$ and $u_i^s(\underset{\sim}{x}, t)$ are the separate contributions from dilatational and shear waves, respectively. The far-field representations of $u_i^d(\underset{\sim}{x}, t)$ and $u_i^s(\underset{\sim}{x}, t)$ are as follows

$$u_i^d = \frac{1}{4\pi\rho r c_d^3}\gamma_i\gamma_j\gamma_k\int_v \dot{m}_{jk}(\underset{\sim}{\xi}, t - r/c_d + \underset{\sim}{\gamma}\cdot\underset{\sim}{\xi}/c_d)\,d^3\underset{\sim}{\xi}$$

$$\begin{aligned} u_i^s = {} & \frac{1}{4\pi\rho r c_s^3}\left[\frac{1}{2}(\delta_{ij}\gamma_k + \delta_{ik}\gamma_j) - \gamma_i\gamma_j\gamma_k\right] \\ & \times \int_v \dot{m}_{jk}(\underset{\sim}{\xi}, t - r/c_s + \underset{\sim}{\gamma}\cdot\underset{\sim}{\xi}/c_s)\,d^3\underset{\sim}{\xi} \end{aligned} \tag{9}$$

where r is the distance from a point in the source region to the observation point, and the integral is over the volume of the source region. The expressions (9) are accurate to terms of order l/r and λ/r where l is a characteristic source dimension and $\lambda = 2\pi c/\omega$ is the wavelength of the emitted radiation associated with the frequency ω.

It is apparent from (9) that the particle velocity $\dot{u}_i$ decays as r^{-1} and, consequently that (3) is satisfied. To verify (4) it is only necessary to show that the left-hand side of (4) vanishes when evaluated from displacements given by (9) since only terms of order r^{-1} have been included in (9). The stresses corresponding to (9) are, to the same order of approximation,

$$\begin{aligned} \sigma_{ij}^d &= -\gamma_k\gamma_l[2\mu\gamma_i\gamma_j + \Lambda\delta_{ij}]I_{kl}(t, r; c_d) \\ \sigma_{ij}^s &= -\mu I_{kl}(t, r; c_s)[\tfrac{1}{2}\gamma_i(\gamma_k\delta_{lj} + \gamma_l\delta_{kj}) \\ &\quad + \tfrac{1}{2}\gamma_j(\gamma_k\delta_{li} + \gamma_l\delta_{ki}) - 2\gamma_i\gamma_j\gamma_k\gamma_l] \end{aligned} \tag{10}$$

where the contributions from u_i^d and u_i^s have been given separately, and

$$I_{kl}(t, r; c) = \frac{1}{4\pi\rho r c^4} \int_v \ddot{m}_{kl}(\underset{\sim}{\xi}, t - r/c + \underset{\sim}{\gamma}\cdot\underset{\sim}{\xi}/c) d^3\underset{\sim}{\xi}. \tag{11}$$

Substitution into the left-hand side of (4) yields

$$\begin{aligned}[\rho c_d^2 - (2\mu + \Lambda)]\gamma_i\gamma_k\gamma_l I_{kl}(t, r; c_d) \\ + [\rho c_s^2 m_i m_l - \mu(\delta_{li} - \gamma_l\gamma_i)]\gamma_k I_{kl}(t, r; c_s)\end{aligned} \tag{12}$$

where the symmetry of I_{kl} [due to the symmetry of $m_{kl}(\underset{\sim}{x}, t)$] has been used. From the definition of $c_d = [(2\mu + \Lambda)/\rho]^{1/2}$, it is immediately clear that the first term, due to the contribution of u_i^d vanishes. That the second vanishes as well can be seen from the definition of $c_s = (\mu/\rho)^{1/2}$ and by noting that $Q_l = \gamma_k I_{kl}$ is a vector with a component in the direction of $\underset{\sim}{m}$ that is given by

$$\underset{\sim}{m}(\underset{\sim}{m}\cdot\underset{\sim}{Q}) = \underset{\sim}{Q} - \underset{\sim}{\gamma}(\underset{\sim}{\gamma}\cdot\underset{\sim}{Q}) \tag{13}$$

where $\underset{\sim}{\gamma}\cdot\underset{\sim}{Q} = Q_p\gamma_p$. The component of $\underset{\sim}{Q}$ orthogonal to both $\underset{\sim}{m}$ and $\underset{\sim}{\gamma}$ vanishes because there is zero shear traction on S in this direction. Thus, the terms of order r^{-1} in the left-hand side of (4) vanish and consequently (2) is a precise expression for the radiated energy in the limit that the surface S recedes to infinity. The radiated energy can be expressed as

$$\begin{aligned}E_R = &\frac{1}{16\pi^2\rho c_d^5}\int_{-\infty}^{\infty}\int_S\left\{\frac{\gamma_j\gamma_k}{r}\int_v \ddot{m}_{jk}(\underset{\sim}{\xi}, t - r/c_d + \underset{\sim}{\gamma}\cdot\underset{\sim}{\xi}/c_d)\, d^3\underset{\sim}{\xi}\right\}^2 dS\, dt \\ &+ \frac{1}{16\pi^2\rho c_s^5}\int_{-\infty}^{\infty}\int_S\left\{\frac{1}{2}\frac{(m_j\gamma_k + m_k\gamma_j)}{r}\int_v \ddot{m}_{jk}(\underset{\sim}{\xi}, t - r/c_s + \underset{\sim}{\gamma}\cdot\underset{\sim}{\xi}/c_s)\, d^3\underset{\sim}{\xi}\right\}^2 dS\, dt.\end{aligned} \tag{14}$$

Also, if the Fourier transform of a function $g(t)$ is defined by

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\, dt, \tag{15}$$

then the Parseval identity

$$\int_{-\infty}^{\infty} |\tilde{g}(\omega)|^2\, d\omega = 2\pi\int_{-\infty}^{\infty} [g(t)]^2\, dt \tag{16}$$

can be used to express E_R in terms of integrals of the spectral density of m_{jk}. [Boatright (1980) has recently used this representation to advantage in a study of spectral theory for circular seismic sources.]

It should be emphasized that the equivalence of (1) and (2) does not require that the source be a "point source", i.e., that the dimensions of the source be much smaller than the wavelengths of any waves emitted by the source ($l \ll \lambda$). Rather, it is required only that the distance between S and the source be large compared to source dimensions and wavelengths ($r \gg l$, $r \gg \lambda$). The operations indicated by (2) seem possible to perform observationally (assuming, of course, that propagation effects due to departure of earth from a homogeneous elastic medium can be handled and neglecting effects due to finite boundaries): particle velocities can be measured in the far field and integrated to get E_R. However, as demonstrated by Kostrov (1968) (also see Rice, 1980) in the case of slip on a planar fault, the complete spatial variation of slip over the fault plane cannot be resolved from far-field observations. Evidently, the shorter wavelength information necessary to resolve completely the fault slip and not transmitted to the far field does not contribute to E_R. Nevertheless, it is not possible to express the radiated energy in terms of the seismic moment tensor, which is defined by

$$M_{ij}(t) = \int_v m_{ij}(\underset{\sim}{\xi}, t)\, d^3\underset{\sim}{\xi} \tag{17}$$

If, however, attention is restricted to wavelengths which are large by comparison to source dimensions (i.e., sufficiently low frequencies) the source is perceived in the far field as a point and the radiated energy can be expressed in terms of M_{ij}.

Point Source Representation

As shown by Kostrov (1970), the point source representation corresponds to neglecting the time differences $\underset{\sim}{\gamma} \cdot \underset{\sim}{\xi}/c$ between different points in the source region. In this case the integrals in (9) simply define the time derivative of the seismic moment tensor (17) and (9) can be rewritten as

$$\begin{aligned} u_i^d &= \frac{1}{4\pi\rho c_d^3 r} \gamma_i\gamma_k\gamma_l \dot{M}_{kl}(t - r/c_d) \\ u_i^s &= \frac{1}{4\pi\rho c_s^3 r}\left[\frac{1}{2}(\gamma_l\delta_{ki} + \gamma_k\delta_{li}) - \gamma_i\gamma_k\gamma_l\right]\dot{M}_{kl}(t - r/c_s). \end{aligned} \tag{18}$$

Substituting these expressions into (2) yields

$$\begin{aligned} E_R = \frac{1}{16\pi^2\rho c_s^5}\int_{-\infty}^{\infty}\int_\Omega \Bigg\{ & \left(\frac{c_s}{c_d}\right)^5 [\gamma_k\gamma_l\ddot{M}_{kl}(t - r/c_d)]^2 \\ & + \left[\frac{1}{2}(\gamma_k m_l + \gamma_l m_k)\ddot{M}_{kl}(t - r/c_s)\right]^2\Bigg\}\, d\Omega\, dt \end{aligned} \tag{19}$$

where dS has been replaced by $r^2 d\Omega$. The second square bracket can be rewritten by using the relation

$$\tfrac{1}{2}(\gamma_k m_l + m_k\gamma_l)\ddot{M}_{kl} = [(\gamma_k\ddot{M}_{kl})(\gamma_p\ddot{M}_{pl}) - (\gamma_k\gamma_l\ddot{M}_{kl})^2]^{1/2} \tag{20}$$

which follows from (13) by using the symmetry of $\ddot{M}_{kl}$, setting $\underset{\sim}{Q} = \underset{\sim}{\gamma} \cdot \underset{\sim}{\ddot{M}}$, and taking the magnitude of both sides. Using (20) in (19) yields

$$\begin{aligned} E_R = \frac{1}{16\pi^2\rho c_s^5}\int_{-\infty}^{\infty}\int_\Omega \Bigg\{ & \left(\frac{c_s}{c_d}\right)^5 [\gamma_k\gamma_l\ddot{M}_{kl}(t - r/c_d)]^2 \\ & + \gamma_k\gamma_l\ddot{M}_{kp}(t - r/c_s)\ddot{M}_{lp}(t - r/c_s) - [\gamma_k\gamma_l\ddot{M}_{kl}(t - r/c_s)]^2\Bigg\}\, d\Omega\, dt. \end{aligned} \tag{21}$$

Because the arguments of M_{kl} depend only on time for a fixed radius r, the integrals over the spherical surface involve only the components of the unit normal. These integrals have the form

$$I_{ijkl} = \int_{\Omega} \gamma_i \gamma_j \gamma_k \gamma_l \, d\Omega. \tag{22}$$

The integration is most easily accomplished by recognizing that the result must be an isotropic tensor and, thus, must comprise sums and products of Kronecker delta's

$$I_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 \delta_{ik} \delta_{jl} + c_3 \delta_{il} \delta_{jk} \tag{23}$$

where c_1 to c_3 are constants. Furthermore, because (22) is symmetric with respect to interchange of any two indices, $c_1 = c_2 = c_3 = c$. The constant c can be determined by integration for a particular component, say $I_{1122} = c$ and the result is

$$I_{ijkl} = \frac{4\pi}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \tag{24}$$

Using this result and the corollary,

$$I_{ijkk} = \int_{\Omega} \gamma_i \gamma_j \, d\Omega = \frac{4}{3} \pi \delta_{ij}, \tag{25}$$

in (21) yields

$$60\pi \rho c_s{}^5 E_R = \left(\frac{c_s}{c_d}\right)^5 \int_{-\infty}^{\infty} \left\{ 2\ddot{M}_{ij}(t - r/c_d) \ddot{M}_{ij}(t - r/c_d) + [\ddot{M}_{kk}(t - r/c_d)]^2 \right\} dt$$
$$+ \int_{-\infty}^{\infty} \left\{ 3\ddot{M}_{ij}(t - r/c_s) \ddot{M}_{ij}(t - r/c_s) - [\ddot{M}_{kk}(t - r/c_s)]^2 \right\} dt. \tag{26}$$

This expression can also be rewritten by introducing the deviatoric part of M_{ij}, i.e.,

$$M'_{ij} = M_{ij} - \delta_{ij} M_{kk}/3 \tag{27}$$

so that $tr\underset{\sim}{M'} = M'_{kk} = 0$. Then (26) becomes

$$60\pi \rho c_s{}^5 E_R = \left(\frac{c_s}{c_d}\right)^5 \int_{-\infty}^{\infty} \left\{ 2\ddot{M}'_{ij}(t - r/c_d) \ddot{M}'_{ij}(t - r/c_d) + \frac{5}{3} [\ddot{M}_{kk}(t - r/c_d)]^2 \right\} dt$$
$$+ 3 \int_{-\infty}^{\infty} \ddot{M}'_{ij}(t - r/c_s) \ddot{M}'_{ij}(t - r/c_s) \, dt. \tag{28}$$

If the source disturbance is confined to a surface of displacement discontinuity, as is typically the case in modeling fault rupture, the seismic moment density tensor has the form (Burridge and Knopoff, 1964; Kostrov, 1970; Aki and Richards, 1980, p. 52)

$$m_{ij}(\underset{\sim}{x}, t) = C_{ijkl}(\underset{\sim}{x})n_k(\underset{\sim}{x})\Delta u_l(\underset{\sim}{x}, t)\delta_D(S_F) \tag{29}$$

where $\underset{\sim}{n}$ is the local unit normal to the surface, Δu_l is the displacement discontinuity, and $\delta_D(S_F)$ is a Dirac singular function which converts an integral over a volume containing S_F to an integral over the surface S_F. In a homogeneous, isotropic elastic solid the modulus tensor is given by (6) and, in this case, the moment density tensor is

$$m_{ij}(\underset{\sim}{x}, t) = [\mu(n_j\Delta u_i + n_i\Delta u_j) + \Lambda n_k\Delta u_k\delta_{ij}]\delta_D(S_F). \tag{30}$$

For shear faulting $n_k\Delta u_k = 0$ and if the fault surface is assumed to be flat and to have a normal in the x_2 direction,

$$m_{ij}(\underset{\sim}{x}, t) = \mu(\delta_{2j}\Delta u_i + \delta_{2i}\Delta u_j)\delta_D(S_F). \tag{31}$$

The seismic moment (17) is

$$M_{ij}(t) = \mu\int_{S_F}[\delta_{2j}\Delta u_i(\xi_1, \xi_3, t) + \delta_{2i}\Delta u_j(\xi_1, \xi_3, t)]\, d\xi_1\, d\xi_3. \tag{32}$$

If it is further assumed that slip is only in the x_1 direction, $\Delta u_i = \Delta u\delta_{i1}$, and $M_{12} = M_{21} = M$ are the only nonzero components of the seismic moment tensor

$$M(t) = \mu\int_{S_F}\Delta u(\xi_1, \xi_3, t)\, d\xi_1\, d\xi_3. \tag{33}$$

The expression for the energy radiated by a point source (28) then reduces to

$$30\pi\rho c_s^5 E_R = 2\left(\frac{c_s}{c_d}\right)^5\int_{-\infty}^{\infty}\ddot{M}^2(t - r/c_d)\, dt + 3\int_{-\infty}^{\infty}\ddot{M}^2(t - r/c_s)\, dt. \tag{34}$$

Alternatively, E_R can be expressed in terms of the Fourier transform of M by using (15) and (16).

Fault Surface Representation

If the source can be idealized as a propagating crack, then, as shown by Kostrov (1974) the radiated energy can be expressed in terms of the fault surface traction and particle velocity. The resulting form for E_R is useful for assessing the physical sources of radiated energy. Let S_f be a surface coinciding with the fault surfaces (excluding the extending edges), S_o, a tube enclosing the crack edges and $\underset{\sim}{n}$, the unit normal directed into the material bounded by S, S_f, and S_o. (Hence $\underset{\sim}{n} = -\underset{\sim}{\gamma}$ on S and $\underset{\sim}{n}$ is the outward normal to the crack surfaces on S_f.) A general power balance for this material requires that

$$-\int_S\bar{\sigma}_{ij}n_j\dot{u}_i\, dS - \dot{K} - \dot{U} = \bar{F}(t) + \int_{S_F}\bar{\sigma}_{ij}n_j\dot{u}_i\, dS \tag{35}$$

where $\dot{K}$ and $\dot{U}$ are the rates of increase of kinetic energy and strain energy, respectively; and $\bar{F}(t)$ is the time rate of energy flow to the extending fault edges. The stress $\bar{\sigma}_{ij}$ is the total stress, i.e.,

$$\bar{\sigma}_{ij} = \sigma_{ij} + \bar{\sigma}^{\circ}_{ij} \tag{36}$$

and the total displacement is

$$\bar{u}_i = u_i + \bar{u}_i^{\circ}$$

with $\dot{\bar{u}}_i^{\circ} = 0$. The left-hand side of (35) is the excess of the total power input over the increase in total internal energy. The right-hand side can be regarded as dissipation due to energy flux to the extending fault edges (first term) and dissipation due to the work of the fault surface tractions (second term). More precisely, the energy flux to the extending edges is defined as (Freund, 1972a; Kostrov, 1974)

$$\bar{F}(t) = \lim_{S_o \to 0} \int_{S_o} \left\{ \bar{\sigma}_{ij} n_j \dot{u}_i + \left[\frac{1}{2} \bar{\sigma}_{ij} \bar{u}_{i,j} + \frac{1}{2} \dot{u}_i \dot{u}_i \right] v \right\} dS$$

where v is the normal velocity of the fault edge in its own plane. The contribution from $\bar{F}(t)$ arises because the stress and particle velocity are singular at the edge of an extending crack in a linear elastic solid. Although this singularity is sometimes regarded as unrealistic, the energy flux can be given a simple physical interpretation as the negative of the work of unloading the fracture surfaces, or the energy which is needed to overcome the fracture resistance of the material just ahead of the fault edge (Freund, 1972a; Kostrov, 1974). Specific forms for $\bar{F}(t)$ have been given for semi-infinite cracks (or finite cracks prior to reflected waves) by Freund (1972b) for plane strain extension (mode I), by Fossum and Freund (1975) for plane strain shear (mode II) and by Atkinson and Eshelby (1968) for antiplane shear (mode III). Because the initial stress is nonsingular at a point on the fault edge for any nonzero amount of fault expansion at that point, the energy flux into the edge depends only on the incremental stress and displacement, i.e., $\bar{F}(t)$ may be replaced by $F(t)$ in all cases.

Equation (35) can be rearranged by substituting (36) and using the definition of radiated energy (1). The result is

$$-\dot{E}_R - \int_S \bar{\sigma}^{\circ}_{ij} n_j \dot{u}_i \, dS - \dot{K} - \dot{U} = \bar{F}(t) + \int_{S_F} \bar{\sigma}_{ij} n_j \dot{u}_i \, dS.$$

Integrating in time and recognizing that the kinetic energy vanishes in the initial and final static states yield the following expression

$$E_R + \int_S \bar{\sigma}^{\circ}_{ij} n_j u_i^{\text{final}} \, dS + (\bar{U}_{\text{final}} - \bar{U}_{\text{initial}})$$
$$= -\int_{-\infty}^{\infty} \bar{F}(t) \, dt - \int_{-\infty}^{\infty} \int_{S_F} \bar{\sigma}_{ij} n_j \dot{u}_i \, dS \, dt. \tag{37}$$

For quasi-static propagation $E_R = 0$ and (37) reduces to a formula for the change in strain energy

$$\bar{U}_{\text{final}} - \bar{U}_{\text{initial}} + \int_S \bar{\sigma}^{\circ}_{ij} n_j u_i^{\text{final}}\, dS = -\int_{-\infty}^{\infty} \int_{S_F} [\bar{\sigma}_{ij} n_j \dot{u}_i]_o\, dS\, dt - \int_{-\infty}^{\infty} \bar{F}_o(t)\, dt \quad (38)$$

where the subscript "o" denotes the value during quasi-static extension. The integral over S is frequently neglected by assuming S coincides with the earth's surface and, hence, is traction-free. However, this assumption is not needed to obtain the following expressions for radiated energy. Subtracting (38) from (37) yields

$$E_R = \int_{-\infty}^{\infty} \int_{S_F} \{[n_j \bar{\sigma}_{ij} \dot{u}_i]_o - [n_j \bar{\sigma}_{ij} \dot{u}_i]\}\, dS\, dt + \int_{-\infty}^{\infty} \{\bar{F}_o(t) - \bar{F}(t)\}\, dt \quad (39)$$

where the superposed bars can be omitted in this equation. The integrand of the second term is the difference in energy flux to the crack-tip during quasi-static and dynamic propagation of the fault edge and the integrand of the first term is the difference in the work rate of the fault surface tractions (behind the fault edge) during quasi-static and dynamic extension. If the fault surface tractions are bounded, the first term of (39) can be integrated by parts. The result is

$$E_R = \int_{-\infty}^{\infty} \int_{S_F} \{[n_j \dot{\bar{\sigma}}_{ij} u_i] - [n_j \dot{\bar{\sigma}}_{ij} u_i]_o\}\, dS\, dt + \int_{-\infty}^{\infty} [\bar{F}_o(t) - \bar{F}(t)]\, dt \quad (40)$$

where the integrated terms vanish because the initial and final static states are fixed and the superposed bars can again be omitted. This expression is essentially that given by Kostrov [1974, equation (2.26)], although he does not write it in this way. This form for E_R makes clear the role of the time-dependent surface tractions in radiating energy.

An alternative form for the radiated energy can be obtained by using (36) in the first term on the right-hand side of (38). The result is

$$\int_{-\infty}^{\infty} \int_{S_F} [\bar{\sigma}_{ij} n_j \dot{u}_i]_o\, dS\, dt = \int_{-\infty}^{\infty} \int_{S_F} [\sigma_{ij} n_j \dot{u}_i]_o\, dS\, dt + \int_{S_F} \bar{\sigma}^{\circ}_{ij} n_j u_i^{\text{final}}\, dS. \quad (41)$$

If σ_{ij} depends on time only through u_i, i.e.,

$$\sigma_{ij}(\underset{\sim}{x}, t) = \sigma_{ij}[\underset{\sim}{x}, u_i(\underset{\sim}{x}, t)],$$

then

$$\int_{-\infty}^{\infty} \int_{S_F} [\sigma_{ij} n_j \dot{u}_i]_o\, dS\, dt = \int_{S_F} [(\sigma_{ij} n_j)_{\text{ave}} u_i^{\text{final}}]\, dS \quad (42)$$

where this equation can be used to define the average traction $(\sigma_{ij} n_j)_{\text{ave}}$. In the special case where $\sigma_{ij}n_j$ is proportional to u_i

$$(\sigma_{ij}n_j)_{\text{ave}} = \tfrac{1}{2}[(\sigma_{ij}n_j)_{\text{final}} + (\sigma_{ij}n_j)_{\text{initial}}]. \tag{43}$$

Note that $(\sigma_{ij}n_j)_{\text{initial}}$ is not the traction existing ahead of the extending fault edge, but rather that which exists on the fault surface just after the passage of the edge. Noting that

$$({}^{\circ}{}_{ij}n_j)_{\text{ave}} = (\bar{\sigma}_{ij}n_j)_{\text{ave}} - \bar{\sigma}^{\circ}_{ij}n_j$$

and using (42) in (41) yields

$$\int_{-\infty}^{\infty}\int_{S_F} [\bar{\sigma}_{ij}n_j\dot{u}_i]_o \, dS \, dt = \int_{-\infty}^{\infty}\int_{S_F} [(\bar{\sigma}_{ij}n_j)_{\text{ave}}\dot{u}_i]_o \, dS \, dt.$$

Consequently, (38) can be written as follows

$$\bar{U}_{\text{final}} - \bar{U}_{\text{initial}} + \int_S \bar{\sigma}^{\circ}_{ij}n_j u_i^{\text{final}} \, dS = \int_{-\infty}^{\infty}\int_{S_F} [(\bar{\sigma}_{ij}n_j)_{\text{ave}}\dot{u}_i]_o \, dS \, dt - \int_{-\infty}^{\infty} \bar{F}_o(t) \, dt. \tag{44}$$

If $\bar{F}_o$ and the integral over S are assumed to be negligible, this equation reduces to the usual formula for strain energy change (e.g., Dahlen, 1974; Aki and Richards, 1980, p. 57). Subtracting (44) from (37) reveals that an alternative expression for the radiated energy is

$$E_R = \int_{-\infty}^{\infty}\int_{S_F} \{[(\bar{\sigma}_{ij}n_j)_{\text{ave}}\dot{u}_i]_o - [\bar{\sigma}_{ij}n_j\dot{u}_i]\} \, dS \, dt + \int_{-\infty}^{\infty} \{\bar{F}_o(t) - \bar{F}(t)\} \, dt \tag{45}$$

where again the superposed bars can be omitted.

If the fault surface tractions are time independent (except possibly in a region near the fault edge which is small compared with the overall length), the first term of (39), (40), and (45) vanishes and the radiated energy is simply

$$E_R = \int_{-\infty}^{\infty} [\bar{F}_o(t) - \bar{F}(t)] \, dt. \tag{46}$$

The expression for the radiated energy used by Husseini and Randall (1976) is a special case of (46). When the crack propagation speeds are near the limiting velocities (Rayleigh wave speed in plane strain or shear wave speed in antiplane strain), as is typical for observed earthquake faults, $\bar{F}(t)$ is small compared with $\bar{F}_o(t)$. Thus, in this case, the radiated energy is approximately equal to

$$E_R = \int_{-\infty}^{\infty} \bar{F}_o(t) \, dt. \tag{47}$$

The meaning of this equation can be clarified by returning to (38) and recognizing that the first term on the right-hand side is the work done by the fault surface tractions during quasi-static fault extension. In the simplest models of shear faults, these tractions are assumed to be equal to a uniform value of the friction stress and the corresponding work done is taken to be consumed in the generation of heat (e.g., Orowan, 1960). Thus, the right-hand side of (47) is the excess of the strain energy change over the quasi-static work of the fault surface tractions at least if the integral over S in (38) can be neglected. In the case of a shear fault with uniform stress drop, Kanamori (1977) has called this excess W_o and has argued on an empirical basis that for large earthquakes W_o is a good approximation to the radiated energy.

If the stress near the fault edge is not idealized as singular, the terms $\bar{F}(t)$ and $\bar{F}_o(t)$ vanish and the entire contribution to E_R is from the first term of (39), (40), and (45). This is typically the case in dislocation models for which the rise time is finite (e.g., Haskell, 1964). Hence, the radiated energy is due to the deviation of the actual fault surface tractions during the dynamic event from the average fault plane tractions. It is interesting to note that for the configuration considered by Haskell (1964) (introduction of a rectangular dislocation loop of constant relative displacement), the strain energy change $U_{\text{final}} - U_{\text{initial}}$ is unbounded but the radiated energy [which Haskell computed by using (2)] is finite. Because $\bar{F}(t)$ and $\bar{\sigma}_{ij}^{\circ}$ are zero, the remaining term in (37) must also be unbounded. However, the sum of this term and strain energy change must be finite in order that E_R is finite.

Discussion

Various expressions for the energy radiated by seismic sources in linear elastic solids have been examined for the purpose of clarifying the meaning of radiated energy and its relation to seismic parameters. The term "earthquake energy" often seems to be applied either to E_R, the radiated energy, or to W_o, the excess of the strain energy change over the work done by the fault surface tractions during quasi-static fault extension [see (38) and (47)]. However, these two quantities are equal only in special cases. As indicated by (40), E_R depends on the time dependence of the fault surface tractions and particle velocities and, in particular, on the difference between the values during the actual dynamic process and those which would occur if the same fault extension occurred quasi-statically. Hence, as pointed out by Kostrov (1974), the radiated energy cannot, in general, be determined from differences in the static end states. Nevertheless, E_R can be determined by the measurement and integration of far-field particle velocities (2) even though this information is not sufficient to resolve the spatial distribution of fault slip. If, however, the fault surface tractions are assumed to be time independent, as in the case of many simple fault models, and the fault rupture speed is such that the second term in (46) is small, then E_R can be approximated as in (47), where the right-hand side of this equation is the quantity Kanamori (1977) has labeled W_o. This quantity does depend only on the difference between static end states.

Kanamori (1977) has argued that for large earthquakes, W_o is a good approximation to the energy computed from the Gutenberg-Richter energy magnitude relationship ($E = 1.5\ M + 11.8$ where M is the magnitude). More specifically, Kanamori (1977) showed that a magnitude computed using W_o as the energy in the Gutenberg-Richter relation agrees well with the surface wave magnitude. The Gutenberg-Richter relationship (Gutenberg and Richter, 1942) was apparently established by estimating empirically the integral in (2) and hence the energy in this relation is an estimate for E_R. However, it is not clear whether Kanamori's (1977) results should be construed as evidence that the conditions for which (47) applies (time-independent fault surface tractions and rupture speed near the limiting value) are satisfied for large earthquakes. Certainly, it seems unlikely that fault surface tractions would be time independent during faulting but it is possible that the contribution to the radiated energy is small.

Although the dislocation model of Haskell (1964) is often used as a basis for discussing energy radiation in earthquakes, it would seem to have some deficiencies in this regard. [Related deficiencies of the Haskell model have been discussed by Madariaga (1978).] As mentioned earlier, the strain energy change for this configuration is infinite and hence the ratio of radiated energy to strain energy change is inevitably zero. A configuration for which the strain energy change is finite would be more useful for studying the relation between W_o and E_R.

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