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A FORMULATION FOR STUDYING COUPLED DEFORMATION PORE FLUID DIFFUSION EFFECTS ON LOCALIZATION OF DEFORMATION

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ABSTRACT

A formulation is presented for studying the effects of coupling between deformation and pore fluid diffusion on the development of localized deformation in fissured rock masses. Specifically, deformation of a solid containing an initial imperfection in the form of a planar band of weakened material is considered. Deformation in the band and in the material outside is assumed to be homogeneous and the fluid mass flux out of the band is assumed to be proportional to the difference between the pore fluid pressure in the band and that in the material outside. Although a rate-independent model is assumed for the material behavior, coupling of deformation with diffusion introduces rate dependence into the problem. The linearized equations predict that the difference between the strain-rate of the material inside and outside of the band grows exponentially in time when the conditions for localization of deformation are met in terms of the drained (long-time) response of the band material. However, this strain-rate difference does not become unbounded until conditions for localization of deformation are met in terms of the undrained (short-time) response of the band material. For dilatant materials in which an increase in confining pressure inhibits inelastic response, coupling between deformation and diffusion results in a delay in the onset of failure. However, failure, as predicted in terms of the response of the band material, can occur well before corresponding failure conditions are met in terms of the material outside the band.

NOMENCLATURE

$\underset{\sim}{a}$	vector denoting the right hand side of (19)
$\underset{\sim}{A}$	inverse of $\underset{\sim}{N}\cdot\underset{\sim}{K}^b\cdot\underset{\sim}{N}$
$\underset{\sim}{B}$	2nd order modulus tensor
c	diffusivity
$\underset{\sim}{F}$	deformation gradient tensor
h	width of the weakened band of material
$\underset{\sim}{I}$	identity tensor
J	= det $\underset{\sim}{F}$, Jacobian of deformation from $\underset{\sim}{X}$ to $\underset{\sim}{x}$

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$\underset{\sim}{k}$	matrix of co-factors of $\underset{\sim}{N}\cdot\underset{\sim}{K}^b\cdot\underset{\sim}{N}$
K	4th order constitutive tensor for drained response
$\overline{K}$	4th order constitutive tensor for undrained response
m	fluid mass per unit volume of porous solid
$\underset{\sim}{N}$	unit normal in reference configuration
p	pore fluid pressure
$\underset{\sim}{Q}$	difference between $\dot{\underset{\sim}{F}}\cdot\underset{\sim}{N}$ in band and in material outside
$\underset{\sim}{R}$	2nd order constitutive tensor
r	diagonal components of $\underset{\sim}{R}$ for incrementally isotropic response
$\underset{\sim}{s}$	nominal stress tensor
$\overline{\underset{\sim}{s}}$	$= \underset{\sim}{s} + p\underset{\sim}{I}$, effective stress tensor
t	time
$\underset{\sim}{x}$	position vector in current configuration
$\underset{\sim}{X}$	position vector in reference configuration
$\dot{\gamma}_0$	imposed constant strain-rate
δ_{ij}	Kronecker delta, $= 1$ if $i=j$, $= 0$ if $i\neq j$
$\Delta(\ldots)$	difference between $(\ldots)$ in the band and in the material outside
κ	$= c/(h^2\dot{\gamma}_0)$
τ	$= \dot{\gamma}_0 t$, nondimensional time
ρ	mass density of homogeneous pore fluid
$\underset{\sim}{\sigma}$	Cauchy (true) stress tensor
η	scalar compliance relating $\dot{p}$ to $\dot{m}/\rho$.

Superscripts

b	denotes value in weakened band
o	denotes value in material outside band

INTRODUCTION

Localized deformation is a common and, perhaps, prevalent mode of failure in geological materials. Shear bands in overconsolidated clay slopes and earthquake faults are examples. Axisymmetric compression tests on rock samples at pressures and temperatures typical of the earth's crust generally indicate that final failure of the sample is preceded by the development of localized deformation in a fault or gouge zone. Although most earthquakes occur along established fault zones, at least some events break relatively fresh or largely rehealed rock. Because the seismic signals from both types of events can be modelled as due to slip on a planar discontinuities, for those that break relatively intact rock localization of deformation along a fault plane presumably precedes or coincides with final failure.

Rudnicki and Rice (1) have analyzed conditions for the development of localized deformation, described as a bifurcation from homogeneous deformation, for a constitutive law intended to model the response of brittle rock. The presence of an infiltrating pore fluid can, however, affect the inception and development of localized deformation. The most direct effect of the pore fluid is to reduce the effective compressive hydrostatic stress by some fraction of the pore fluid pressure. Brace and Martin (2) have studied this phenomenon, in the failure of laboratory rock samples. An additional and, perhaps, more important effect of the pore fluid is the rate-dependence introduced into the response by the coupling of deformation with pore fluid diffusion. This rate dependence influences not only the onset of failure but also its rate of development. Martin (3) has recently studied this effect in laboratory deformation of Westerly granite.

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Coupling of pore fluid diffusion with inelastic deformation arises from the tendency of brittle rock to dilate, or increase the volume of pore space, when sheared in the inelastic range (e.g., 4). When dilation occurs more rapidly than fluid mass can diffuse into the newly created pore space, the pore fluid pressure is reduced and the effective compresssive stress, the difference between the compressive stress and the pore fluid pressure, is increased. This increase inhibits further inelastic deformation due to microcracking and frictional sliding and the rock mass is said to be dilatantly hardened. Rice (5) has analyzed the stability of dilatant hardening for the special deformation state of simple shear. He found that dilatantly hardened homogeneous deformation became unstable, in the sense that infinitesimal nonuniformities grew exponentially in time, when the conditions for localization of deformation (1) are met in terms of the underlying drained, or constant pore fluid pressure, response. Rice (5) suggested, however, that dilatant hardening as it may occur in situ, would be modelled more closely by considering shear of a rock mass with a pre-existing imperfection or weakened zone. In this case, the growth of the nonuniformity would depend on the relative rates of pore fluid diffusion and imposed deformation. Of particular interest is the time delay between the point at which small nonuniformities begin to grow in time and final failure.

Rudnicki (6) has recently adopted Rice's suggestion and studied in detail the shear of a layer containing an imperfection in the form of a narrow sublayer with material properties slightly different from those of the surrounding material. More specifically, the sublayer is assumed to have a peak stress that is lower than that in the adjacent material. The deformation in the sublayer and in the surrounding material is assumed to be homogeneous and the fluid mass flux between the sublayer and the surrounding material is assumed to be proportional to the difference in the pore fluid pressures.

This paper extends the approach used in Rudnicki (6) to arbitrary deformation states. This work has elements in common with other studies of the development of shear localization from planar imperfections (7,8,9) and, particularly, with such studies applied to rate-dependent solids (10,11). The motivation for this work is, as already mentioned, to examine the effects of coupling between deformation and pore fluid diffusion on the development of localized shear deformation. Nevertheless, the approach used here may also be useful for studying localization in other situations where coupling between deformation and diffusion of another species or heat is important.

The paper begins by reviewing the conditions for localization of deformation. Then, these conditions are applied to a constitutive law appropriate for fluid-infiltrated solids. Differences between the resulting equations and those for non-fluid infiltrated solids are discussed.

CONDITIONS FOR LOCALIZATION OF DEFORMATION

Rice (12) has given a general review of localization of deformation and this section follows closely his description. The position of a material particle in some reference configuration is denoted by $\underset{\sim}{X}$ and the current position of the same particle is $\underset{\sim}{x}$. The deformation is described by the deformation gradient tensor $\underset{\sim}{F}$ which is given by

$$\underset{\sim}{F} = \partial \underset{\sim}{x} / \partial \underset{\sim}{X} \quad \text{or} \quad F_{ij} = \partial x_i / \partial X_j \tag{1}$$

where the second form applies for components in a rectangular cartesian coordinate system. Stress is measured by the nonsymmetric nominal stress $\underset{\sim}{s}$. The Cauchy (true) stress $\underset{\sim}{\sigma}$ is related to $\underset{\sim}{s}$ by (e.g., 13)

$$\underset{\sim}{\sigma} = J^{-1} \underset{\sim}{F} \cdot \underset{\sim}{s} \quad \text{or} \quad \sigma_{ij} = J^{-1} F_{ik} s_{kj} \tag{2}$$

where $J = \det(\underset{\sim}{F})$ is the ratio of the volume of a material element in the current configuration to its volume in the reference configuration. Quasi-static equilibrium in the absence of body forces is expressed by

$$\partial s_{ij} / \partial X_i = 0 \tag{3}$$

Consider a solid that is homogeneous except for the presence of an initial imperfection in the form of a planar band with normal $\underset{\sim}{N}$ in the reference state. The material properties in this band are assumed to be different from those in the surrounding material. The solid is subjected to a deformation which, in the absence of the imperfection, would give rise to the uniform deformation increment $\dot{\underset{\sim}{F}}^o$, where the dot denotes the time derivative. The goal is to determine the resulting deformation in the band subject to the requirements of compatibility and continuing equilibrium. Compatibility requires that the displacement increments be continuous at the interface of the band and the adjacent material. As a consequence, $\dot{\underset{\sim}{F}}$ in the band must have the following form (12):

$$\dot{\underset{\sim}{F}}^b = \dot{\underset{\sim}{F}}^o + \dot{\underset{\sim}{Q}}\underset{\sim}{N} \quad \text{or} \quad \dot{F}^b_{ij} = \dot{F}^o_{ij} + \dot{Q}_i N_j \tag{4}$$

where $\dot{\underset{\sim}{Q}}$ may, in general, depend on the distance across the band, but vanishes outside the band. If, as will be done here, the band is assumed to be sufficiently narrow so that deformation there is homogeneous, $\dot{\underset{\sim}{Q}}$ does not depend on position within the band. The compatibility requirement (4) can be rewritten as

$$\Delta\dot{\underset{\sim}{F}} = \dot{\underset{\sim}{Q}}\underset{\sim}{N} \tag{5}$$

where $\Delta(\ldots) = (\ldots)^b - (\ldots.)^o$. Continuing equilibrium requires that (12)

$$\underset{\sim}{N}\cdot\dot{\underset{\sim}{s}}^b = \underset{\sim}{N}\cdot\dot{\underset{\sim}{s}}^o \quad \text{or} \quad \underset{\sim}{N}\cdot\Delta\dot{\underset{\sim}{s}} = 0 \tag{6}$$

At this point, it is necessary to specify a constitutive law relating $\dot{\underset{\sim}{s}}$ to $\dot{\underset{\sim}{F}}$. A class of rate-independent solids can be described by a relation of the form

$$\dot{\underset{\sim}{s}} = \underset{\sim}{K}:\dot{\underset{\sim}{F}} \quad \text{or} \quad \dot{s}_{ij} = K_{ijk\ell}\dot{F}_{k\ell} \tag{7}$$

where the constitutive tensor $\underset{\sim}{K}$ may have different branches for loading and unloading but is otherwise independent of $\dot{\underset{\sim}{F}}$. More specifically, the relation (7) is appropriate for elastic-plastic materials with smooth yield surfaces. If the material is one for which inelastic strain increments can be assumed normal to the yield surface, then the moduli in (7) satisfy the symmetry

$$K_{ijk\ell} = K_{\ell kji} \tag{8}$$

However, for rocks and soils, the assumption of normality is generally too restrictive and the condition (8) is not satisfied (1,14).

If (7) is substituted into (6) and (4) is used, the result can be written

$$(N_i K^b_{ijk\ell} N_\ell)\dot{Q}_k = N_i(\underset{\sim}{K}^o - \underset{\sim}{K}^b)_{ijk\ell}\dot{F}^o_{k\ell} \tag{9}$$

where the superscripts "b" and "o" on the constitutive moduli designate values in the band and in the material outside. This equation is identical to equation (5) of Rice (12). When the material is assumed to be homogeneous, $\underset{\sim}{K}^o = \underset{\sim}{K}^b \equiv \underset{\sim}{K}$ and (9) becomes

$$(N_i K_{ijk\ell} N_\ell)\dot{Q}_k = 0 \tag{10}$$

A non-trivial solution for the $\dot{Q}_k$ is possible only when

$$\det(N_i K_{ijk\ell} N_\ell) = 0 \tag{11}$$

and localization occurs at the first point in a program of loading when the condition (11) is met. If the material in the band is slightly different from the surrounding material, (9) can, in principle, be solved for the $\dot{Q}_k$ as a function of the deformation rate $\dot{F}^o$ in the material outside the band. This procedure has been used in studies of the effects of voids on shear localization in porous metals (e.g., 8,9). The deformation rate in the band becomes large relative to that in the material outside the band when

$$\det(N_i K^b_{ijk\ell} N_\ell)\dot{Q}_k = 0 \tag{12}$$

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Plausible values for κ can be obtained by using values of c, h, and $\dot{\gamma}_0$ appropriate to field and laboratory conditions. Rice and Simons (21) have suggested values of c in the range of 0.1 to 1.0 m^2/s are representative of field conditions near faults. Values of h in the range 1 m to 10 m might be appropriate for the widths of fault zones, and $\dot{\gamma}_0 = 10^{-14}/s$ is representative of the tectonic strain rate in southern California (22). For these values κ is in the range 10^{11} to 10^{14}. For laboratory conditions, $c=10^{-4}m^2/s$, $h=10^{-1}$ to $10^{-2}m$ and $\dot{\gamma}_0=10^{-6}/s$ to $10^{-8}/s$ may be appropriate and these yield κ in the range 10^6-10^8. Thus, for the applications considered here, κ is a large parameter. When (26) is rewritten in terms of the nondimensional time τ, κ multiplies Δp. Consequently, the solution of (26) will yield $p^b \approx p^0$ until NK^bN becomes small enough so that its product with κ is comparable to the other terms in the equation. The presence of the large parameter κ requires that some care be used in the numerical evaluation of (19) and (25).

CONCLUDING DISCUSSION

The framework described here generalizes to arbitrary deformation states an approach used by Rudnicki (6) to study dilatant hardening effects on the development of localized deformation in simple shear. Because axisymmetric compression is the most common laboratory testing configuration, implementation of this framework may make possible more detailed comparison of predictions with laboratory observations. Unfortunately, Martin's (3) study is the only one of which this author is aware that examines coupled-deformation diffusion effects on the development of failure. However, Wong (personal communication, 1982) has recently suggested that the stabilizing effects of coupling between deformation and pore fluid diffusion may be exploited to study post-peak deformation in laboratory rock samples and this application may provide added impetus for laboratory study of these effects.

Studies of localization of deformation on non-fluid infiltrated solids (1) predict that plane strain deformation states are much more favorable for localization than axisymmetric deformation states. The formulation described here can also be used to investigate the effects of coupled deformation diffusion on this preference for localized deformation to develop in plane strain.

A central assumption in the formulation here is that the fluid mass flux out of the weakened band is proportional to the difference in the pore fluid pressures. A more typical approach would use Darcy's law (19,20) which, in the absence of body forces, states that the fluid mass flux is proportional to the negative of the gradient in the pore fluid pressure. Combining Darcy's law with an equation expressing fluid mass conservation and a constitutive law like (16) leads to a nonhomogeneous, nonlinear diffusion equation for the pore fluid pressure. Unfortunately, this approach leads to fully nonhomogeneous deformation and, hence, is incompatible with the assumption of homogeneous deformation within the band. Crude asymptotic analysis (23) suggests the development of severe gradients at the interface between the layers as conditions for localization are approached. Consequently, even solution by numerical methods poses serious obstacles and is inevitably expensive. Given the present uncertainty about material parameters, investment in such an approach would seem at this time to be unwise. If the deformation in the band is required to be homogeneous, then it is possible to derive (22) by considering the average flux out of the band and using the equation of fluid mass conservation. Nevertheless, it would be useful to have a better understanding of the effect of this approximation.

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and localization is said to occur at this point. As noted by Rice (12), this condition may be met well before the corresponding condition in terms of the moduli of the material outside the band.

Needleman and Rice (15) have emphasized that an entirely equivalent formulation can be given in terms of the Cauchy (true) stress $\boldsymbol{\sigma}$ and the symmetric part of the velocity gradient tensor. They note, however, that the formulation in terms of $\dot{\mathbf{s}}$ and $\dot{\mathbf{F}}$ seems more suitable for the following the development of localized deformation from an initial imperfection.

The next section describes the form of a constitutive law for a fluid-infiltrated solid.

CONSTITUTIVE LAW FOR A FLUID-INFILTRATED SOLID

The formulation of a constitutive law for a fluid infiltrated solid requires the introduction of an additional "deformation" variable m, the fluid mass per unit reference volume of porous solid, and an additional "stress" variable p, the pore fluid pressure. The pore fluid pressure is defined as the pressure in an idealized reservoir of homogeneous pore fluid that would prevent any fluid mass flux if the reservoir were connected to an element of the fluid-infiltrated solid. An incrementally linear form for the constitutive law, analogous to (7), is the following:

$$\dot{\mathbf{s}} = \overline{\mathbf{K}} : \dot{\mathbf{F}} - \eta^{-1}\mathbf{B}(\dot{m}/\rho) \tag{13}$$

$$\eta\dot{p} = (\dot{m}/\rho) - \mathbf{R} : \dot{\mathbf{F}} \tag{14}$$

where ρ is the mass density of the pore fluid; $\overline{\mathbf{K}}$, $\mathbf{B}$, and $\mathbf{R}$ are modulus tensors and η is a scalar coefficient. The constitutive parameters may, again, have different branches for loading and unloading but are otherwise independent of $\dot{\mathbf{F}}$ and $\dot{m}$. The deformation gradient $\mathbf{F}$ is that associated with deformation of the solid and $\mathbf{N} \cdot \mathbf{s}$ is the total force per unit reference area of porous solid with normal $\mathbf{N}$. It will be convenient to rewrite (13) and (14) by solving (14) for $\dot{m}/\rho$ and substituting into (13). The result is

$$\dot{\mathbf{s}} = \mathbf{K} : \dot{\mathbf{F}} - \mathbf{B}\dot{p} \tag{15}$$

$$\dot{m}/\rho = \mathbf{R} : \dot{\mathbf{F}} + \eta\dot{p} \tag{16}$$

where $\mathbf{K} = \overline{\mathbf{K}} - \eta^{-1}\mathbf{B}\mathbf{R}$.

When the deformation is so slow that any changes in pore fluid pressure can be equilibrated by fluid mass flux, $\dot{p}=0$ and conditions are said to be drained. Thus, the tensor of incremental moduli $\mathbf{K}$ are those governing drained response. The contrasting limit in which deformation is too rapid to allow time for fluid mass flux out of material elements is called undrained response. Setting $\dot{m}=0$ in (13) reveals that $\overline{\mathbf{K}}$ is the tensor of incremental moduli governing undrained response. The rate-of-change of pore fluid pressure during undrained response is obtained from (14) as

$$\dot{p} = -\eta^{-1}\mathbf{R} : \dot{\mathbf{F}} \tag{17}$$

For incrementally isotropic response, the tensor $\mathbf{R}$ has the form

$$\mathbf{R} = r\mathbf{I}$$

where $\mathbf{I}$ is the identity tensor with components δ_{ij} and $r>0$. Thus, for undrained response, $\dot{p}$ is proportional to the negative of the rate of volume straining, $\mathrm{tr}\dot{\mathbf{F}} = \dot{F}_{kk}$. Brittle rocks and overconsolidated soils tend to dilate, or increase volume, when sheared in the inelastic range. If the shearing is rapid enough to approach undrained conditions, the pore fluid pressure tends to decrease. Because a decrease in p increases the effective compressive stress, the total compressive stress minus the pore fluid pressure, and inhibits further inelastic deformation resulting from frictional microscale mechanisms, the undrained response is stiffer than the drained response for these materials.

However, for materials that compact when sheared inelastically, the undrained response will be softer than the drained response.

Some further understanding of (15) and (16) can be gained by considering the special case of incrementally isotropic response and separately incompressible solid and fluid constituents. In this case, $\eta=0$ and the tensors $\mathbf{B}$ and $\mathbf{R}$ reduce to the identity tensors. Thus, $\dot{m}/\rho$ is simply equal to the rate of volume straining. The pore fluid pressure p enters (15) in the manner appropriate to the classical effective stress principle (e.g., 15). This principle states that the mechanical effect of the pore fluid can be incorporated by replacing the stress $\mathbf{s}$ by the effective stress which, in this case, is

$$\bar{\mathbf{s}} = \mathbf{s} + p\mathbf{I}. \qquad (18)$$

For compressible constituents, Rice (17) has shown that this is also the form of the effective stress for inelastic deformation resulting from the extension of sharp-tipped fissures and from frictional sliding on surfaces with small contact areas. More generally, the appropriate form of the effective stress will differ from (18) for elastic deformation (18,19) and for other types of inelastic deformation.

LOCALIZATION OF DEFORMATION IN A FLUID-INFILTRATED SOLID

If the constitutive law described in the previous section (15) is substituted into (6) and (4) is used, the result is

$$(N_i K^b_{ijk\ell} N_\ell)\dot{Q}_k - N_i B^b_{ij}(\dot{p}^b - \dot{p}^o) = N_i(\mathbf{K}^o - \mathbf{K}^b)_{ijk\ell}\dot{F}^o_{k\ell} - N_i(\mathbf{B}^o - \mathbf{B}^b)_{ij}\dot{p}^o \qquad (19)$$

This equation is analogous to (9) but the pore fluid pressure in the band enters as an additional quantity to be determined. The deformation rate and pore fluid pressure outside the band are considered to be prescribed. If the deformation is drained, the pore fluid pressure in the band is equal to that in the surrounding material, and the left-hand side of (19) becomes identical to (9). Consequently, the condition for localization is given by (12). On the other hand, if the deformation in the band is constrained to be undrained, (19) becomes

$$(N_i \bar{K}^b_{ijk\ell} N_\ell)\dot{Q}_k = N_i(\mathbf{K}^o - \mathbf{K}^b)_{ijk\ell}\dot{F}^o_{k\ell} - N_i B^o_{ij}\dot{p}^o \qquad (20)$$

where $\bar{\mathbf{K}}^b$ is the tensor of undrained moduli in the band. Thus, the condition for localization is that

$$\det(N_i \bar{K}^b_{ijk\ell} N_\ell) = 0 \qquad (21)$$

For materials in which the undrained response is stiffer than the drained response, this condition (21) will not be met until after (12). Thus, as shown in other studies (5,20), coupling between deformation and diffusion can delay the onset of failure. Note, however, that the condition (21) will be met before the corresponding condition based on the undrained moduli of the material outside the band.

Analysis of cases other than the limiting ones of drained and undrained deformation requires specification of a relation for the rate at which fluid mass is transferred between the band and the surrounding material. Here, it is simply assumed that the fluid mass flux between the two layers is proportional to the difference in pore fluid pressures in the layers. This assumption, together with the assumption of homogeneous deformation in the band, requires that the rate-of-change of fluid mass per unit volume of the band $\dot{m}^b$ have the following form:

$$\dot{m}^b = -\rho(c/h^2)(p^b - p^o) \qquad (22)$$

where the constant of proportionality has been written as the ratio of a diffusity c to the square of the width of the band h. Combining (22) with (16) and using (4) yields

$$\eta^b(\dot{p}^b - \dot{p}^o) + (c/h^2)(p^b - p^o) + R^b_{ij}N_j\dot{Q}_i = -R^b_{ij}\dot{F}^o_{ji} - \eta^b\dot{p}^o . \qquad (23)$$

Because the $\dot{Q}_i$ depend on the difference $\dot{p}^b - \dot{p}^o$ from (19), it is convenient to eliminate $\dot{Q}_i$ from (23). Except at the point where (12) is satisfied, (19) can be solved for the $\dot{Q}_k$. The result can be expressed as

$$\dot{\mathbf{Q}} = \mathbf{A} \cdot \{\mathbf{N}\cdot\mathbf{B}^b(\dot{p}^b - \dot{p}^o) + \mathbf{N}\cdot(\mathbf{K}^o - \mathbf{K}^b):\dot{\mathbf{F}}^o - \mathbf{N}\cdot(\mathbf{B}^o - \mathbf{B}^b)\dot{p}^o\} \qquad (24)$$

where $\mathbf{A} = (\mathbf{N}\cdot\mathbf{K}^b\cdot\mathbf{N})^{-1}$. Substituting (24) into (23) yields

$$[\eta^b + (\mathbf{R}^b\cdot\mathbf{N})\cdot\mathbf{A}\cdot(\mathbf{N}\cdot\mathbf{B}^b)]\Delta\dot{p} + (c/h^2)\Delta p = -(\mathbf{R}\cdot\mathbf{N})\cdot\mathbf{A}\cdot\mathbf{a} - \mathbf{R}^b:\dot{\mathbf{F}}^o - \eta\dot{p}^o \qquad (25)$$

where $\Delta p = p^b - p^o$ and

$$\mathbf{a} = \mathbf{N}\cdot(\mathbf{K} - \mathbf{K}^b):\dot{\mathbf{F}}^o - \mathbf{N}\cdot(\mathbf{B}^o - \mathbf{B}^b)\dot{p}^o$$

Equation (25) is a nonlinear first order ordinary differential equation for Δp, which is coupled to (23) through the dependence of the constitutive parameters on pore pressure and deformation in the band. The forcing term on the right-hand side involves the difference between the constitutive parameters in the band and the material outside and the imposed deformation and pore fluid pressure.

The matrix $(\mathbf{N}\cdot\mathbf{K}^b\cdot\mathbf{N})$ can formally be inverted by writing

$$(\mathbf{N}\cdot\mathbf{K}^b\cdot\mathbf{N})^{-1} = \mathbf{k}/(NK^bN)$$

where $\mathbf{k}$ is the matrix of cofactors and $NK^bN = \det(\mathbf{N}\cdot\mathbf{K}^b\cdot\mathbf{N})$. Equation (25) can then be rearranged into the following form

$$[\eta^b(NK^bN) + (\mathbf{R}^b\cdot\mathbf{N})\cdot\mathbf{k}\cdot(\mathbf{N}\cdot\mathbf{B}^b)]\Delta\dot{p} + (c/h^2)(NK^bN)\Delta p =$$
$$-(\mathbf{R}^b\cdot\mathbf{N})\cdot\mathbf{k}\cdot\mathbf{a} - (\mathbf{R}^b:\dot{\mathbf{F}}^o + \eta\dot{p}^o)(NK^bN) \qquad (26)$$

If (26) is linearized by regarding the constitutive parameters as fixed at the current values, the result is a linear first order equation with constant coefficients. The homogeneous solutions of this equation exhibit exponential growth or decay in time depending on whether NK^bK is positive or negative. Because this sign changes from positive to negative when the condition for localization (12) is met in terms of the drained moduli of the band, exponential growth of the difference in pore fluid pressure and, hence, the $\dot{Q}_k$ occur at this point. This result is consistent with the conclusion of Rice's (5) analysis for the special deformation state of combined pure shear and uniaxial compression. Note, however, that final failure, corresponding to the possibility of one of the $\dot{Q}_k$ becoming unbounded, does not occur until the condition (21) is met in terms of the undrained moduli of the band. Of particular interest is the time needed to reach this point once the localization condition (12) has been met in terms of the underlying drained response. This time depends on the relative rates of imposed deformation and fluid mass flux between the layers.

If the deformation and pore fluid pressure in the material outside the band are imposed at a constant rate, it is convenient to introduce the nondimensional time

$$\tau = \dot{\gamma}_0 t$$

where $\dot{\gamma}_0$ is a measure of the amplitude of the imposed (constant) strain-rate. Rewriting (19) and (25) in terms of τ makes no change in (19) but causes the coefficient of Δp in (25) to become

$$\kappa = c/h^2\dot{\gamma}_0$$

This parameter is the ratio of the time-scale of imposed deformation to that for diffusion over a length scale of h. For $\kappa\to\infty$, the pore fluid pressure is the same in both layers. For $\kappa\to 0$ there is no fluid mass flux out of the band and, consequently, the deformation there is undrained.