

Plane Strain Dislocations in Linear Elastic Diffusive Solids

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Solutions are obtained for the stress and pore pressure due to sudden introduction of plane strain dislocations in a linear elastic, fluid-infiltrated, Biot, solid. Previous solutions have required that the pore fluid pressure and its gradient be continuous. Consequently, the antisymmetry (symmetry) of the pore pressure p about $y = 0$ requires that this plane be permeable ($p = 0$) for a shear dislocation and impermeable ($\partial p / \partial y = 0$) for an opening dislocation. Here Fourier and Laplace transforms are used to obtain the stress and pore pressure due to sudden introduction of a shear dislocation on an impermeable plane and an opening dislocation on a permeable plane. The pore pressure is discontinuous on $y = 0$ for the shear dislocation and its gradient is discontinuous on $y = 0$ for the opening dislocation. The time-dependence of the traction induced on $y = 0$ is identical for shear and opening dislocations on an impermeable plane, but differs significantly from that for dislocations on a permeable plane. More specifically, the traction on an impermeable plane does not decay monotonically from its short-time (undrained) value as it does on a permeable plane; instead, it first increases to a peak in excess of the short-time value by about 20 percent of the difference between the short and long time values. Differences also occur in the distribution of stresses and pore pressure depending on whether the dislocations are emplaced on permeable or impermeable planes.

Introduction

The presence of an infiltrating fluid that can diffuse in response to an inhomogeneous mean stress field can introduce time-dependence into the response of an otherwise linear-elastic solid. Although a linear theory is obviously an approximation to actual behavior, this theory is rich enough to provide insight into the nature of coupling between deformation and diffusion and guidance into more complicated nonlinear problems. Moreover, there is often insufficient data to warrant the construction of a more elaborate theory.

The equations describing the response of a linear elastic, diffusive solid were first formulated by Biot (1941a) within the context of a fluid-saturated porous elastic solid. More recently, Rice and Cleary (1976) reformulated these equations in a way that is often more convenient. Solutions to these equations have been widely used in consolidation theory (e.g. Biot, 1941b; Biot and Clingan, 1941) and, more recently, in studying the role of coupling between deformation and diffusion of ground water on earth faulting (see Rudnicki, 1985, for a review). The equations have also been applied to biological materials (e.g., Kuei, 1977; Mow and Lai, 1980). Indeed, the formulation is sufficiently general to describe the linearized response of any solid containing a diffusing species that can be

characterized by a relation between two scalar variables, for example, pressure and fractional volume change in the case of groundwater. The formal analogy of these equations to completely coupled thermoelasticity has also been noted (Biot, 1956; Rice and Cleary, 1976; Rice 1979).

This paper considers the problem of plane strain (edge) dislocations in a linear elastic diffusive solid. Booker (1974), using the stress function formulation of McNamee and Gibson (1960a,b) and integral transforms, obtained the solution for a shear (gliding edge) dislocation in the special case that both solid and fluid constituents are incompressible. Rice and Cleary (1976), using a complex variable formulation, derived the solution for arbitrarily compressible constituents. These solutions correspond to the case in which the glide plane of the dislocation (the plane containing the dislocation line and the Burger's vector) is permeable to the diffusing species. Although neither author emphasizes this feature, it results because the mean stress and pore pressure are antisymmetric about the glide plane. If the pore pressure is continuous, then it must be zero on the glide plane. Another possibility, however, is that glide plane is impermeable to the diffusing species. Now the pore pressure can be discontinuous on this plane.

The stresses and displacements for an opening (climbing edge) dislocation can also be obtained from the results of Rice and Cleary (1976) although they do not explicitly display this solution. In this case the boundary condition on the pore fluid pressure corresponds to no flow across the plane containing the Burger's vector and the dislocation line. Again, however,

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there is another possibility: no change in pore fluid pressure on this plane. Now, the symmetry of the mean-stress and pore pressure requires that the gradient of the pore fluid pressure be discontinuous.

In this paper, Fourier and Laplace transforms are used to derive the stresses and pore fluid pressure due to sudden introduction of a plane strain shear dislocation on an impermeable plane and of a plane strain opening dislocation on a permeable plane. These solutions are compared and contrasted with those obtained by Rice and Cleary (1976). Although applications of these solutions are not explored here, Rudnicki (1986) has discussed the implications of the shear dislocation solutions for slip on an impermeable fault in the earth's crust. The dislocation solutions provide only crude models of sliding or opening cracks, but solutions for more realistic geometries can be constructed by superposition or by implementing the fundamental dislocation solutions in a numerical procedure.

This paper first concisely describes the governing equations and obtains the solution for the doubly transformed stresses and pore pressure. Then the boundary conditions for the different solutions are presented. The inversion of the transformed solution for the shear dislocation is discussed in detail, but inversion of the opening dislocation is similar and, consequently, is only outlined. Finally, the interrelations of these solutions with those obtained by Rice and Cleary (1976) are discussed.

Governing Equations

The governing equations for linear elastic, fluid-infiltrated solids were first derived by Biot (1941a), but the description here follows a convenient rearrangement of these equations by Rice and Cleary (1976). In this theory, the presence of the diffusing species is incorporated via two variables in addition to the usual ones of linear elasticity. Here, these are taken to be the pore fluid pressure p and the mass content of diffusing species per unit volume of porous solid m . For plane strain deformation in the xy plane (no displacement in the z direction) the displacements in the x and y directions, u_x and u_y , do not depend on z . The nonzero strains are

$$\epsilon_{\alpha\beta} = \frac{1}{2} (\partial u_\alpha / \partial x_\beta + \partial u_\beta / \partial x_\alpha) \quad (1)$$

where $(\alpha, \beta) = (x, y)$. These strains and the alteration of m from an ambient value m_o are related to the total stresses σ_{xx} , σ_{xy} , and σ_{yy} and to the pore fluid pressure p as follows:

$$2G\epsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu(\sigma_{xx} + \sigma_{yy})\delta_{\alpha\beta} + [3(\nu_u - \nu)/B(1 + \nu_u)]p \delta_{\alpha\beta} \quad (2)$$

$$m - m_o = \frac{3\rho_o(\nu_u - \nu)}{2GB(1 + \nu_u)} [\sigma_{xx} + \sigma_{yy} + 3p/B(1 + \nu_u)] \quad (3)$$

In equations (2) and (3) G is the shear modulus; ν and ν_u are Poisson's ratios governing drained (long-time) and undrained (short-time) response, respectively; B is Skempton's coefficient, the ratio of an increment of pore fluid pressure to an increment of mean normal compression during undrained response; ρ_o is the density of the homogeneous diffusing species; and $\delta_{\alpha\beta}$ is the Kronecker delta ($\delta_{\alpha\beta} = 1$, if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$, otherwise).

For deformation that is slow enough so that any alterations in pore fluid pressure are equilibrated by mass diffusion, the response is said to be drained and, since $p = 0$ in this case, equation (2) reduces to the usual elasticity relation. Deformation that is too rapid to allow time for diffusion is said to be undrained. In this case, $m = m_o$, and solving for p in equation (3) and substituting in equation (2) again yields the form of the usual elasticity relation with ν_u replacing ν .

The final constitutive equation is Darcy's law which, in the absence of body forces, states that the mass flow rate in the α

direction per unit area, q_α , is proportional to the gradient of pore fluid pressure:

$$q_\alpha = -\rho_o \kappa \partial p / \partial x_\alpha \quad (4)$$

Here κ is a permeability often expressed as k/μ where k has the units of area and μ is the fluid viscosity.

For plane strain deformation, the governing field equations can be written as follows in terms of the stresses $\sigma_{\alpha\beta}$ and pore pressure p :

$$\partial \sigma_{xx} / \partial x + \partial \sigma_{xy} / \partial y = 0 \quad (5)$$

$$\partial \sigma_{xy} / \partial x + \partial \sigma_{yy} / \partial y = 0 \quad (6)$$

$$\nabla^2 (\sigma_{xx} + \sigma_{yy} + 2\eta p) = 0 \quad (7)$$

$$(c \nabla^2 - \partial / \partial t) [\sigma_{xx} + \sigma_{yy} + (2\eta/\mu)p] = 0 \quad (8)$$

where $\nabla^2 (\dots) = [(\partial^2 / \partial x^2) + (\partial^2 / \partial y^2)] (\dots)$, c is a diffusivity, $\mu = (\nu_u - \nu) / (1 - \nu)$

and

$$\eta = 3(\nu_u - \nu) / 2B(1 + \nu_u)(1 - \nu).$$

Equations (5) and (6) express equilibrium of total stresses in the absence of body forces and equation (7) expresses compatibility of strains. The diffusion equation (8) is the result of combining Darcy's law (4) with an equation of fluid mass conservation and using equation (7). Comparing equation (8) with (3) reveals that the quantity in square brackets in equation (8) is proportional to the alteration of fluid mass content. Hence, as emphasized by Rice and Cleary (1976), the fluid mass content m satisfies a homogeneous diffusion equation although the pore fluid pressure, in general, does not. Rice and Cleary (1976) have given a full discussion of these equations and have tabulated values of material parameters inferred from laboratory tests on rocks (also see Rudnicki, 1985) and Rice (1979b) and Rice and Rudnicki (1979) have given some estimates of ν and ν_u for conditions near faults in the earth's crust.

The equations (5)–(8), subject to boundary conditions to be discussed in succeeding subsections, will be solved using the Fourier transform on x and the Laplace transform on t . The Laplace transform of a function $f(x, t)$ is defined by

$$\tilde{f}(x, s) = \int_0^\infty \exp(-st) f(x, t) dt \quad (9)$$

and the inversion is denoted by

$$f(x, t) = L^{-1} \{ \tilde{f}(x, s) \} = \frac{1}{2\pi i} \int_{Br} \tilde{f}(x, s) \exp(st) ds \quad (10)$$

where $i = (-1)^{1/2}$ and Br denotes the Bromwich contour. The Fourier transform is defined by

$$\hat{f}(\kappa, s) = \int_{-\infty}^\infty \tilde{f}(x, s) \exp(-i\kappa x) dx \quad (11)$$

with inversion

$$\tilde{f}(x, s) = F^{-1} [\hat{f}(\kappa, s)] = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(\kappa, s) \exp(i\kappa x) d\kappa \quad (12)$$

Applying the Fourier and Laplace transforms to equations (5)–(8) yields the following results:

$$i\kappa \hat{\sigma}_{xx} + \frac{d\hat{\sigma}_{xy}}{dy} = 0 \quad (13)$$

$$i\kappa \hat{\sigma}_{xy} + \frac{d\hat{\sigma}_{yy}}{dy} = 0 \quad (14)$$

$$\left(-\kappa^2 + \frac{d^2}{dy^2} \right) (\hat{\sigma}_{xx} + \hat{\sigma}_{yy} + 2\eta \hat{p}) = 0 \quad (15)$$

$$\left\{ \frac{d^2}{dy^2} - n^2(\kappa) \right\} [\hat{\sigma}_{xx} + \hat{\sigma}_{yy} + (2\eta/\mu)\hat{p}] = 0 \quad (16)$$

where $n^2(\kappa) = \kappa^2 + s/c$. These equations are identical to those obtained by Rice and Simons (1976) except that $-\kappa V/c$ in their expression for $n^2(\kappa)$ is replaced by s/c . Consequently, the solution of these equations can be obtained directly from their results and is as follows:

$$\frac{1}{2}(\hat{\sigma}_{xx} + \hat{\sigma}_{yy}) = A e^{-m(\kappa)y} + B e^{-n(\kappa)y} \quad (17)$$

$$\eta \hat{p} = -\mu A e^{-m(\kappa)y} - B e^{-n(\kappa)y} \quad (18)$$

$$\frac{1}{2}(\hat{\sigma}_{yy} - \hat{\sigma}_{xx}) = [C + m(\kappa)y A] e^{-m(\kappa)y} - \frac{B}{s/c} [\kappa^2 + n^2(\kappa)] e^{-n(\kappa)y} \quad (19)$$

$$\hat{\sigma}_{xy} = -[\mu \kappa m^{-1}(\kappa) C + \mu \kappa y A] e^{-m(\kappa)y} + \frac{B}{s/c} 2\mu \kappa n(\kappa) e^{-n(\kappa)y} \quad (20)$$

where A , B , and C are functions of κ and s to be determined, and $m^2(\kappa) = \kappa^2$. Note that the doubly transformed solution for the fluid mass content per unit volume m is proportional to $B e^{-n(\kappa)y}$. To insure convergence of the inversion integrals in $y > 0$, $m(\kappa)$ and $n(\kappa)$ are subject to the following restrictions:

$$\text{Re}[m(\kappa)] \geq 0 \quad (21)$$

$$\text{Re}[n(\kappa)] \geq 0 \quad (22)$$

where $\text{Re}[\dots]$ stands for "the real part of $[\dots]$."

The functions A , B , and C can be determined from the boundary conditions which are discussed in the next subsection.

Boundary Conditions

The introduction of a shear (gliding edge) dislocation at the origin corresponds to cutting the negative x axis, displacing the top to the right and the bottom to the left by the same amount, then bonding the cut elastic plane back together. The resulting discontinuity in the x displacement is described as follows:

$$u_x(x, y=0^+, t) - u_x(x, y=0^-, t) = [2\pi(1 - \nu_u) b_x / G] H(-x) H(t) \quad (23)$$

where $H(\dots)$ denotes the unit step function and the notation $y = 0^\pm$ indicates that u_x is to be evaluated as the x axis is approached from above or below. The magnitude of the discontinuity is measured by b_x and the factor $2\pi(1 - \nu_u)/G$ has been introduced with a view to simplifying later expressions. Because the displacements are antisymmetric with respect to the plane $y = 0$, the problem can be formulated in the upper half-plane, $y \geq 0$, with equation (23) rewritten as

$$u_x(x, 0^+, t) = [\pi(1 - \nu_u) b_x / G] H(-x) H(t) \quad (24)$$

Because of antisymmetry and continuity of total tractions on $y = 0$, the normal stress on this plane σ_{yy} is zero:

$$\sigma_{yy}(x, 0^+, t) = 0 \quad (25)$$

If the pore fluid pressure p is continuous, then antisymmetry requires that it be zero on the plane $y = 0$:

$$p(x, 0^+, t) = 0 \quad (26)$$

This is the problem for which the solution has been given by Rice and Cleary (1976) (and earlier by Booker, 1974, for incompressible constituents corresponding to $B = 1$ and $\nu_u = 0.5$). Because $\partial p / \partial y$ is not zero on $y = 0$ in this case, flow across $y = 0$ occurs according to equation (4). Another possibility is, however, that the plane $y = 0$ is impermeable to the diffusing species. As discussed by Rudnicki (1986), this can occur for an earth fault because clay gouge or finely ground material is present in the fault zone. In this case no flow can occur across $y = 0$ and the boundary condition enforcing this constraint is the following:

$$\frac{\partial p}{\partial y}(x, 0^+, t) = 0 \quad (27)$$

Because the solution to the field equations is written in terms of stresses, it is also convenient to express the boundary condition (24) in terms of the stresses. Differentiating (24) with respect to x yields

$$\frac{\partial u_x}{\partial x}(x, 0^+, t) = -[\pi(1 - \nu_u) b_x / G] \delta(x) H(t) \quad (28)$$

where $\delta(x)$ is the Dirac delta function. Because $\epsilon_{xx} = \partial u_x / \partial x$, equation (28) can be substituted into equation (2) and the result, after using equation (25), is

$$-2\pi(1 - \mu) b_x \delta(x) H(t) = \sigma_{xx}(x, 0, t) + 2\eta p(x, 0, t) \quad (29)$$

where μ and η are defined following equation (8). If the plane $y = 0$ is permeable and equation (26) is satisfied, the second term vanishes. In this case, the change in fluid mass content on $y = 0$ is proportional to $\sigma_{xx}(x, 0, t)$. Because m satisfies the homogeneous diffusion equation, the solution is that for a fluid mass dipole (Carslaw and Jaeger, 1959) given by Rice and Cleary (1976). If the fault plane is impermeable and equation (27) is appropriate, the resulting boundary condition on m is not so simple and this is a source of the additional complexity in this solution by comparison with that for the permeable plane.

The boundary condition for an opening (climbing edge) dislocation corresponds to introducing a discontinuity in the y displacement on the negative x axis. This problem can again be formulated in the upper half-plane, $y \geq 0$, by noting that the displacements are now symmetric about $y = 0$. The boundary conditions can be written as follows:

$$u_y(x, 0^+, t) = [\pi(1 - \nu_u) b_y / G] H(-x) H(t) \quad (30)$$

$$\sigma_{xy}(x, 0^+, t) = 0 \quad (31)$$

where, again, the constant factor multiplying b_y has been introduced to simplify later expressions.

If the derivative of the pore pressure in the y direction is continuous, then the symmetry of the problem requires that it be zero on $y = 0$. Now, however, an alternative boundary condition is equation (26). In this case the fluid mass flux is discontinuous on $y = 0$. This boundary condition models a thin high permeability layer in which the easy flow of fluid maintains the pore fluid pressure at its ambient value. This boundary condition may also be appropriate when opening is accompanied by injection of fluid mass.

In the next section the solution for the shear dislocation with an impermeable boundary at $y = 0$ (equation (27)) will be completed. The following section treats the opening dislocation with a permeable boundary at $y = 0$ (equation (26)). Because the conversion of the boundary condition (30) to a condition on the stresses is accomplished more easily in terms of the transformed quantities, this task is deferred to this later section.

Shear Dislocation on an Impermeable Boundary

The boundary conditions for the shear dislocation are equa-

tions (25), (29), and, for an impermeable boundary at $y = 0$, equation (27). Taking the Fourier and Laplace transforms of these equations, then substituting equations (17)–(20) yields three equations for the functions A , B , and C . Solving these equations and substituting these expressions into equations (17)–(20) yields the doubly transformed stresses and pore fluid pressure. The expressions for the mean stress and pore pressure are as follows:

$$\hat{\sigma} = (-b_x \pi / s) \{ e^{-m(\kappa)y} - \mu [m(\kappa) / n(\kappa)] e^{-n(\kappa)y} \} \quad (32)$$

$$\eta \hat{p} = (b_x \pi / s) \mu \{ e^{-m(\kappa)y} - [m(\kappa) / n(\kappa)] e^{-n(\kappa)y} \} \quad (33)$$

where $\hat{\sigma} = (1/2)(\hat{\sigma}_{xx} + \hat{\sigma}_{yy})$. It will be convenient to combine equations (19) and (20) into the complex form:

$$\tau = \frac{1}{2} (\sigma_{yy} - \sigma_{xx}) + i \sigma_{xy} \quad (34)$$

The result for the double transform of τ , after substituting the expressions for A , B , and C , is as follows:

$$\begin{aligned} \hat{\tau} = & (b_x \pi / s) [1 + \kappa / m(\kappa)] [1 - m(\kappa)y] e^{-m(\kappa)y} \\ & - (b_x \mu \pi c / s^2) [m(\kappa) / n(\kappa)] \{ [\kappa + n(\kappa)]^2 e^{-n(\kappa)y} \\ & - 2\kappa^2 [1 + \kappa / m(\kappa)] e^{-m(\kappa)y} \} \end{aligned} \quad (35)$$

The Laplace transform variable s appears only in $n(\kappa)$ and as a simple divisor. Terms without $n(\kappa)$ can be inverted immediately by noting that s^{-1} is the transform of the unit step function. These terms give the instantaneous undrained response and it can be anticipated that the spatial dependence, given by the inversion of the Fourier transforms in those terms, is identical to that of the usual elasticity solution. This can be verified by doing the following inverse transforms:

$$F^{-1} \{ e^{-m(\kappa)y} \} = y / \pi r^2 \quad (36)$$

$$F^{-1} \{ [1 + \kappa / m(\kappa)] [1 - m(\kappa)y] \} = i x (x - iy)^2 / \pi r^4 \quad (37)$$

where $r^2 = x^2 + y^2$.

The expression for the mean normal stress (32) and pore fluid pressure (33) can be written using equation (36) as follows:

$$\sigma = -b_x \{ (y/r^2) - \mu I(x, y, t) \} \quad (38)$$

$$\eta p = \mu b_x \{ (y/r^2) - I(x, y, t) \} \quad (39)$$

where the Laplace transform of I is given by

$$\tilde{I} = (2s)^{-1} \int_{-\infty}^{\infty} [m(\kappa) / n(\kappa)] \exp[i\kappa x - n(\kappa)y] d\kappa \quad (40)$$

and equations (38) and (39) are understood to apply for $t \geq 0$. The restrictions on $m(\kappa)$ and $n(\kappa)$ (equations (21), (22)) can be used to convert equation (40) to the following integral over positive values of κ :

$$\tilde{I} = s^{-1} \int_0^{\infty} \kappa (\kappa^2 + s/c)^{-1/2} \cos(\kappa x) \exp[-y(\kappa^2 + s/c)^{1/2}] d\kappa \quad (41)$$

where the expression for $n(\kappa)$ has been used. Substitution of equation (37) into (35) and use of equations (21), (22) leads to an expression for τ :

$$\tau = b_x [i x / (x + iy)^2] - \mu b_x \left\{ i \frac{\partial^2}{\partial x \partial y} I^* - \frac{\partial^2}{\partial x^2} I^* - I^\# + I \right\} \quad (42)$$

where the Laplace transforms of I^* and $I^\#$ are given by

$$\tilde{I}^*(x, y, s) = (2c/s) \tilde{I}(x, y, s) \quad (43)$$

and

$$\tilde{I}^\#(x, y, s) = (2c/s^2) \int_0^{\infty} \kappa^3 (\kappa^2 + s/c)^{-1/2} \exp[i\kappa(x + iy)] d\kappa \quad (44)$$

The inversion of $I(x, y, t)$ is described in the Appendix. The result can be written compactly as

$$I(x, y, t) = -\text{Im} \{ W(x, y, t) / z \} \quad (45)$$

where $z = x + iy$, $\text{Im} [\dots]$ denotes “the imaginary part of [. . .]” and $W(x, y, t)$ is defined by

$$W(x, y, t) = \text{erfc}[y / (4ct)^{1/2}] + \exp(-r^2 / 4ct) \text{erf}[i x / (4ct)^{1/2}] \quad (46)$$

In equation (46) $\text{erf}(\xi)$ is the error function defined by equation (7.1.2) of Abramowitz and Stegun (1964) (hereafter abbreviated AS):

$$\text{erf}(\xi) = (2/\pi^{1/2}) \int_0^\xi \exp(-\alpha^2) d\alpha \quad (47)$$

where ξ can be complex and the complementary error function is given by

$$\text{erfc}(\xi) = 1 - \text{erf}(\xi) \quad (48)$$

The task remaining is the inversion of the integrals $I^\#$ and I^* . Because of equation (43), I^* is given by

$$I^*(x, y, t) = 2c \int_0^t I(x, y, \lambda) d\lambda \quad (49)$$

Substituting equation (45) into (49) yields an expression for I^* :

$$\begin{aligned} I^*(x, y, t) = & (8ct y / r^2) i^2 \text{erfc}[y / (4ct)^{1/2}] \\ & - [2x^2 (4ct)^{1/2} / r^2] i \text{erfc}[y / (4ct)^{1/2}] \\ & + 2|x| \int_{y/r}^1 (1 - \xi^2)^{1/2} \text{erfc}[\xi r / (4ct)^{1/2}] d\xi \end{aligned} \quad (50)$$

where $i^n \text{erfc}(z)$ are repeated integrals of the complementary error function [AS, Section 7.2].

The details of the inversion of equation (44) for $I^\#$ are described in the Appendix and the result is

$$\begin{aligned} I^\#(x, y, t) = & -2(4ct/\pi)^{1/2} z^{-2} - i [4ctz^{-3} \\ & + 2/z] w[z / (4ct)^{1/2}] \end{aligned} \quad (51)$$

where (AS, 7.1.3)

$$w(\zeta) = \exp(-\zeta^2) \text{erfc}(-i\zeta) \quad (52)$$

Substituting equations (45), (50), and (51) into equations (38), (39), and (42) and carrying out the differentiations in equation (42) yield the following expressions for the stresses and pore fluid pressure due to sudden introduction of a shear dislocation on an impermeable plane:

$$\sigma = b_x \text{Im} \{ [1 - \mu W(x, y, t)] / z \} \quad (53)$$

$$\eta p = -\mu b_x \text{Im} \{ [1 - W(x, y, t)] / z \} \quad (54)$$

$$\begin{aligned} \tau = & i b_x / z^2 - \mu b_x \left\{ i 4ctz^{-3} \{ w[z / 4ct]^{1/2} \} - W(x, y, t) \right. \\ & + 2(4ct/\pi)^{1/2} z^{-2} [1 - \exp(-y^2 / 4ct)] + z^{-2} \text{Im} \{ z W(x, y, t) \} \\ & \left. + 2iz^{-1} w[z / 4ct]^{1/2} \right\} \end{aligned} \quad (55)$$

The first term in each expression gives the instantaneous response at $t = 0$. These terms are identical to the usual elasticity expressions with the undrained value of Poisson's ratio, ν_u . For $t \rightarrow \infty$, these expressions again reduce to those of classical elasticity with the drained value of Poisson's ratio.

Opening Dislocation on a Permeable Plane

The solution for an opening dislocation with a permeable boundary ($p = 0$) at $y = 0$ can be obtained in a manner similar to that of the last section and, hence, will be described concisely. As before, the solution to the governing equations (5)–(8) is given by equations (17)–(20) subject to equations (21) and (22), and the functions A , B , and C are to be determined by the boundary conditions (26), (30), and (31). Because the boundary condition (30) is not expressed in terms of stresses and pore pressure some manipulations are, however, required.

Differentiating equation (30) with respect to x and using equation (31) with equations (1) and (2) yields

$$\frac{\partial u_x}{\partial y}(x,0,t) = [\pi(1-\nu_u)b_y/G]\delta(x)H(t) \quad (56)$$

Doubly transforming gives

$$\frac{d\hat{u}_x}{dy}(\kappa,0,s) = \pi(1-\nu_u)b_y/Gs \quad (57)$$

This condition can be converted to one on stress and pore pressure by using equation (1) in (2) with $\alpha = \beta = x$, differentiating with respect to y , and doubly transforming. The result is

$$2G\mu\kappa \frac{d\hat{u}_x}{dy} = \frac{d\hat{\sigma}_{xx}}{dy}(1-\nu) - \nu \frac{d\hat{\sigma}_{yy}}{dy} + 2\eta(1-\nu) \frac{d\hat{p}}{dy} \quad (58)$$

Recognizing that equilibrium (6) and (31) require $d\hat{\sigma}_{yy}/dy$ to vanish on $y = 0$ and substituting from equation (57) yields the desired condition:

$$2\pi \mu b_y(1-\mu)/s = \frac{d\hat{\sigma}_{xx}}{dy}(\kappa,0,s) + 2\eta \frac{d\hat{p}}{dy}(\kappa,0,s) \quad (59)$$

The solutions for A , B , and C in equations (17)–(20) can be determined by doubly transforming (26), (31), and (59), then substituting equations (17)–(20). Solving the resulting three equations for A , B , and C , substituting into equations (17)–(20) and manipulating in the manner of the solution for the shear dislocation yields:

$$\sigma = b_y \{ (x/r^2) - \mu K(x,y,t) \} \quad (60)$$

$$\eta p = -\mu b_y \{ (x/r^2) - K(x,y,t) \} \quad (61)$$

$$\tau = b_y (\nu y/r^2) - \mu b_y \left\{ K(x,y,t) - K^\#(x,y,t) - 2i \frac{\partial^2 K^*}{\partial x \partial y} - 2 \frac{\partial^2 K^*}{\partial y^2} \right\} \quad (62)$$

where the Laplace transforms of K , $K^\#$, and K^* are as follows:

$$\tilde{K}(x,y,t) = s^{-1} \int_0^\infty \exp[-(\kappa^2 + s/c)^{1/2}y] \sin(\kappa x) d\kappa \quad (63)$$

$$\tilde{K}^\#(x,y,t) = (2\mu c/s^2) \int_0^\infty \kappa(\kappa^2 + s/c)^{1/2} \exp[\mu\kappa(x + \nu y)] d\kappa \quad (64)$$

$$\tilde{K}^*(x,y,t) = (c/s)\tilde{K}(x,y,t) \quad (65)$$

Inversion of these expressions proceeds along the same lines as inversion of the corresponding integrals in the shear dislocation solution. The results are as follows:

$$K(x,y,t) = \text{Re} \{ W(x,y,t)/z \} \quad (66)$$

$$K^*(x,y,t) = 2ct(x/r^2) \{ \text{erfc}[y/(4ct)^{1/2}] - 2i^2 \text{erfc}[y/(4ct)^{1/2}] - y \text{sgn}(x) \int_{y/r}^1 (1-u^2)^{1/2} \text{erfc}[ur/(4ct)^{1/2}] du \} \quad (67)$$

$$K^\#(x,y,t) = 4ct z^{-3} w[z/(4ct)^{1/2}] - 2i(4ct/\pi)z^{-2} \quad (68)$$

where the notation is the same as that used earlier. The final expressions for the stress and pore pressure are obtained by substituting equations (66)–(68) into equations (60)–(62) and carrying out the differentiations in equation (62):

$$\sigma = b_y \text{Re} \{ [1 - \mu W(x,y,t)]/z \} \quad (69)$$

$$\eta p = -\mu b_y \text{Re} \{ [1 - W(x,y,t)]/z \} \quad (70)$$

$$\tau = \mu b_y (y/z^2) - \mu b_y \{ 4ctz^{-3} [W(x,y,t) - w(z/(4ct)^{1/2})] + 2i(4ct/\pi)^{1/2} z^{-2} [1 - \exp(-y^2/4ct)] - z^{-2} \text{Re} \{ z W(x,y,t) \} \} \quad (71)$$

These expressions reduce to the usual ones from ordinary elasticity in the limits $t \rightarrow 0$ (undrained response) and $t \rightarrow \infty$ (drained response). In the latter limit, $p = 0$ and the drained Poisson's ratio ν enters; in the former $m = m_o$ and the undrained Poisson's ratio ν_u enters.

Discussion

The similarity between the solutions for the shear dislocation (53)–(55) and the opening dislocation (69)–(71) suggests that they can be combined advantageously in a form analogous to that of complex variable elasticity. To this end, define the complex Burger's vector as

$$b = b_x + ib_y \quad (72)$$

Then the two solutions can be written compactly as follows:

$$\sigma = \text{Im} \{ bz^{-1} [1 - \mu W(x,y,t)] \} \quad (73)$$

$$\eta p = -\mu \text{Im} \{ bz^{-1} [1 - W(x,y,t)] \} \quad (74)$$

$$\tau = \mu z^{-2} \text{Re}(bz) - \mu \{ \mu b_x ctz^{-3} [w(z/(4ct)^{1/2}) - W(x,y,t)] + 2b(4ct/\pi)^{1/2} z^{-2} [1 - \exp(-y^2/4ct)] + z^{-2} \text{Im} \{ b\bar{z} W(x,y,t) \} + 2ib_x w[z/(4ct)^{1/2}] \} \quad (75)$$

where $\bar{b} = b_x - ib_y$. For comparison, the solution for a shear dislocation on a permeable boundary and an opening dislocation on an impermeable boundary (Rice and Cleary, 1976) can be written in the same form:

$$\sigma = \text{Im} \{ bz^{-1} [1 - \mu \exp(-r^2/4ct)] \} \quad (76)$$

$$\eta p = -\mu \text{Im} \{ bz^{-1} [1 - \exp(-r^2/4ct)] \} \quad (77)$$

$$\tau = \mu z^{-2} \text{Re}(bz) - \mu \{ \mu b_x ctz^{-3} [1 - \exp(-r^2/4ct)] + z^{-2} \text{Im} \{ b\bar{z} \exp(-r^2/4ct) \} \} \quad (78)$$

(Rice and Cleary, 1976, display only the solution for the shear dislocation in polar coordinates, but the solution for the opening dislocation is extracted from their results for the complex stress functions).

It is of interest to compare the stresses induced by the dislocations on $y = 0$ for the various cases. For the shear dislocation on the impermeable plane and the opening dislocation on the permeable plane the tractions on $y = 0$ are as follows:

$$\sigma_{yy} + i\sigma_{xy} = (ib_x/x) \{ 1 + \mu(4ct/x^2)[1 - e^{-x^2/4ct}] - 2\mu e^{-x^2/4ct} \} + (b_y/x) \{ 1 - \mu(4ct/x^2)[1 - e^{-x^2/4ct}] \} \quad (79)$$

For comparison the tractions obtained from the Rice and Cleary (1976) solution are

$$\sigma_{yy} + i\sigma_{xy} = (ib_x/x) \{ 1 - \mu(4ct/x^2)[1 - e^{-x^2/4ct}] \} + (b_y/x) \{ 1 + \mu(4ct/x^2)[1 - e^{-x^2/4ct}] - 2\mu e^{-x^2/4ct} \} \quad (80)$$

Note that the spatial dependence of the tractions is the same for the opening dislocation and for the shear dislocation if $y = 0$ is permeable or if $y = 0$ is impermeable. The time dependence of the tractions does, however, depend significantly on whether $y = 0$ is permeable or impermeable. In both cases, the traction decays from a short time limit, corresponding to the usual elasticity expression based on the undrained value of Poisson's ratio, to a long time limit that is smaller by the factor $1 - \mu = (1 - \nu_u)/(1 - \nu)$. The time dependence at intermediate times is shown in Fig. 1, which plots

$$\frac{\sigma_{yy}(x,0,t) - \sigma_{yy}(x,0,\infty)}{\sigma_{yy}(x,0,0) - \sigma_{yy}(x,0,\infty)} \quad (81)$$

against $4ct/x^2$ for the opening dislocation on permeable and impermeable boundaries. As shown, the induced stress for the permeable boundary decays monotonically from the short-

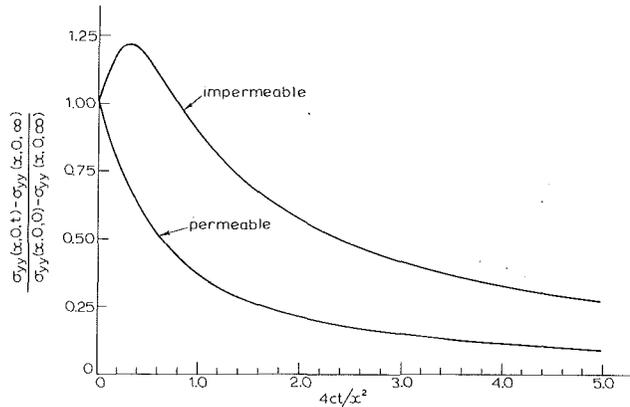


Fig. 1 Time-dependence of the normal traction σ_{yy} on $y = 0$ ahead ($x > 0$) of an opening dislocation at the origin. Results are shown for $y = 0$ permeable and impermeable to the diffusing species. The plot for the shear traction σ_{xy} ahead of a shear dislocation would be identical.

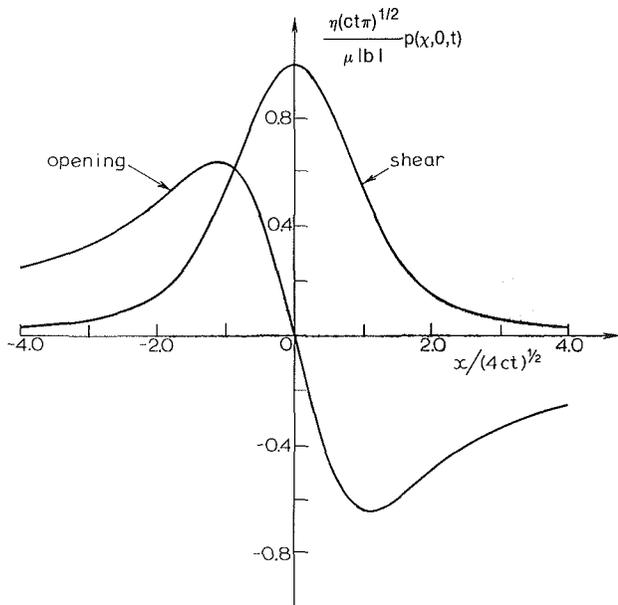


Fig. 2 Nondimensional pore pressure induced on an impermeable plane $y = 0$ by shear and opening dislocations at the origin. The plot is for a fixed time not equal to zero. The pore pressure for the shear dislocation is shown for $y = 0^+$; values for $y = 0^-$ are the negative of those shown.

time undrained value to the long-time drained value. In contrast, the stress on the impermeable boundary first rises to a maximum that exceeds the undrained value by approximately 20 percent of the difference between the undrained and drained values. This maximum occurs at $4ct/x^2 \approx 0.3$. A plot of σ_{xy} on $y = 0$ for the shear dislocation would be identical to Fig. 1.

As discussed by Rudnicki (1986), the increase of the shear stress predicted for the impermeable fault suggests that the effect of coupling between diffusion and deformation is initially destabilizing for sudden seismically emplaced slip. Also the differences in the time scale of shear stress decay for permeable and impermeable faults suggest differences in the effects of coupling on the reloading of faults, which has been proposed as a mechanism for aftershocks, and on processes preceding earthquakes.

Figure 2 shows the pore pressure in nondimensional form $\eta p(ct\pi)^{1/2}/\mu |b|$ induced on $y = 0$ by a shear dislocation and an opening dislocation on an impermeable plane. For the shear dislocation, the pore pressure is antisymmetric about $y = 0$. Consequently, the pore pressure is discontinuous on $y = 0$ and the values on $y = 0^-$ are the negative of those shown in

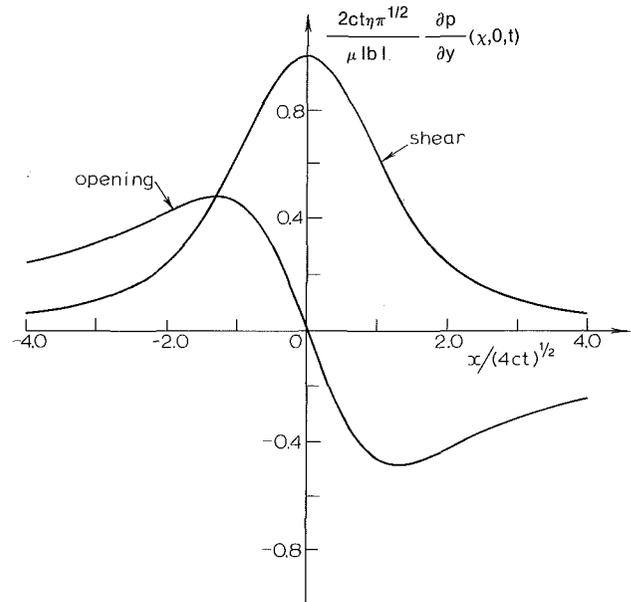


Fig. 3 Nondimensional gradient of pore fluid pressure (proportional to the negative of the fluid mass flux) on a permeable plane $y = 0$ due to shear and opening dislocations at the origin. The plot is for a fixed time not equal to zero. Values shown for the opening dislocation are for $y = 0^+$; those for $y = 0^-$ are the negative of those shown.

Fig. 2. For the opening dislocation, the pore pressure is symmetric about $y = 0$ and, consequently, continuous on $y = 0$.

Figure 3 plots the gradient of pore pressure in nondimensional form $(2ct\eta\pi^{1/2}/\mu |b|)\partial p/\partial y$ on $y = 0$ induced by dislocations on a permeable plane. This quantity is proportional to the negative of the fluid mass flux across $y = 0$ (4). As noted earlier, $\partial p/\partial y$ is antisymmetric about $y = 0$ for the opening dislocation and, hence, is discontinuous on $y = 0$. As shown in Fig. 3, $\partial p/\partial y$ is negative for $x > 0$ and positive for $x < 0$. Consequently, there is a net gain of fluid mass on $y = 0$ for $x < 0$ and a net loss on for $x > 0$. For the shear dislocation, $\partial p/\partial y$ on $y = 0$ is positive and symmetric about $x = 0$. Hence, fluid flows from the upper half-plane to the lower. The nature of the solutions and differences and similarities among them are further illustrated in Figs. 4–9. These figures plot contours of the pore pressure, mean stress, and the magnitude of τ in nondimensional form for the various solutions. These plots are all for a fixed time not equal to zero. Contours for the solutions due to Rice and Cleary (1976), that is, the shear dislocation on a permeable plane and the opening dislocation on an impermeable plane, are shown dashed.

Figures 4 and 5 plot contours of the nondimensional pore pressure $\eta p(4ct)^{1/2}/\mu |b|$ in the upper half plane. Figure 4 shows the contours for the shear dislocation ($b = b_x$) on permeable (dashed lines) and impermeable planes. The values in the lower half-plane are the negative of those shown. The contours coincide for large y , but differ near $y = 0$ because of the different boundary conditions there. As shown, the contours for the shear dislocations on an impermeable plane meet $y = 0$ at right angles as required by the boundary condition. Also, note that the maximum pore pressure change for the impermeable plane occurs at the origin whereas that for the permeable plane occurs at a finite value of y that increases with increasing time. Figure 5 shows the contours of nondimensional pore pressure for the opening dislocation on permeable and impermeable (dashed lines) planes. The pore pressure induced by an opening dislocation on an impermeable plane is identical to that for the shear dislocation on a permeable plane rotated 90 deg counterclockwise. As in the ordinary elasticity solution for shear and opening dislocations this feature applies to the entire stress and pore pressure

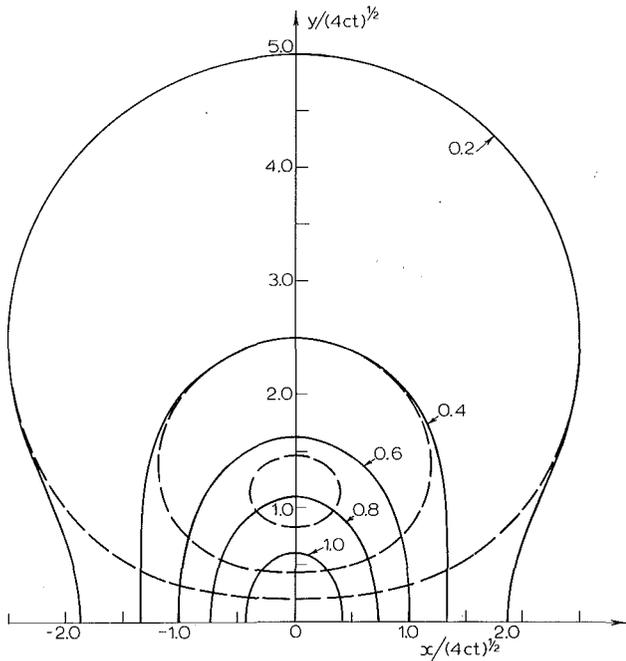


Fig. 4 Contours of nondimensional pore pressure $\eta p(4ct)^{1/2} / \mu |b|$ induced in $y \geq 0$ by a shear dislocation at the origin for the plane $y = 0$ impermeable and permeable (dashed)

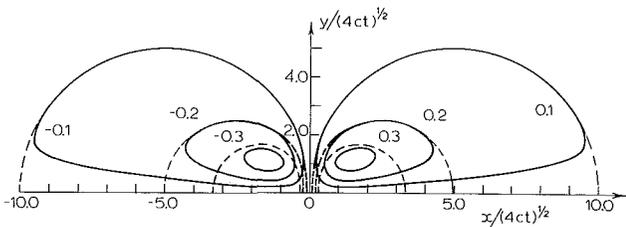


Fig. 5 Same as Fig. 4 for an opening dislocation. Dashed lines indicate the solution when $y = 0$ is impermeable.

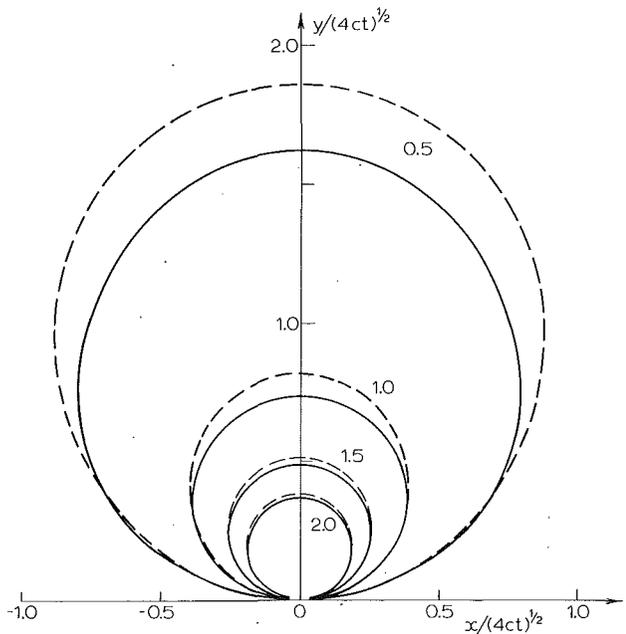


Fig. 6 Contours of nondimensional mean stress $\sigma(4ct)^{1/2} / |b|$ induced in $y \geq 0$ by a shear dislocation at the origin for the plane $y = 0$ impermeable and permeable (dashed)

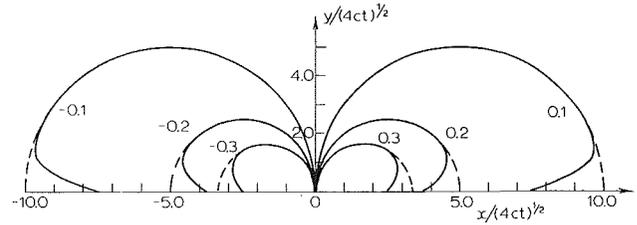


Fig. 7 Same as Fig. 6 for an opening dislocation. Dashed lines indicate the solution when $y = 0$ is impermeable.

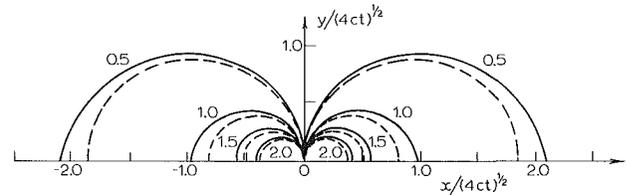


Fig. 8 Contours of nondimensional shear $|\tau|(4ct)^{1/2} / |b|$ induced in $y \geq 0$ by a shear dislocation at the origin for the plane $y = 0$ impermeable and permeable (dashed)

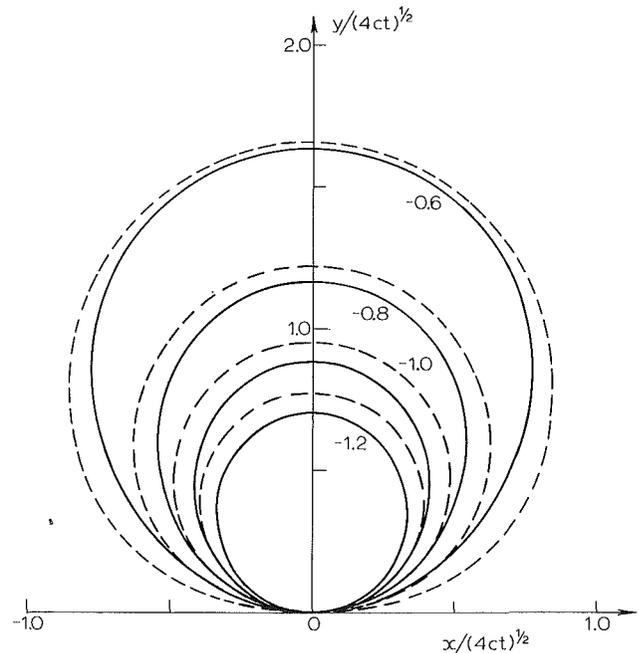


Fig. 9 Same as Fig. 8 for an opening dislocation. Dashed lines indicate the solution when $y = 0$ is impermeable.

fields: those for the opening dislocation on an impermeable plane can be obtained from the shear dislocation on a permeable plane by 90 deg counterclockwise rotation. As is evident from Figs. 4 and 5 the solutions with discontinuous pore pressure or fluid mass flux on $y = 0$ do not possess this property.

Contours of the mean stress, in the nondimensional form $\sigma(4ct)^{1/2} / |b|$ are shown in Fig. 6 for a shear dislocation ($b_x = 1.0$, $b_y = 0$) and in Fig. 7 for an opening dislocation ($b_x = 0$, $b_y = 1.0$) on permeable and impermeable planes. Figures 8 and 9 show contours of the magnitude of the shear stress in the nondimensional form $|\tau|(4ct)^{1/2} / |b|$ for shear (Fig. 8) and opening (Fig. 9) dislocations. In each plot, the two solutions shown approach the undrained solution ($t = 0$) far from the origin and the drained solution ($t \rightarrow \infty$) near the origin. The approach to these limits need not, however, be the same for the two solutions. This is the cause of the different positions

of the contours near the origin in Fig. 6 and for large values of y in Fig. 9.

Acknowledgment

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APPENDIX

This appendix describes some details of the inversion of the integrals $I(x, y, t)$ and $I^\#(x, y, t)$. The Laplace transforms of I and $I^\#$ are given by equations (41) and (44), respectively.

First consider the inversion of $I(x, y, t)$. Interchanging the order of the Laplace and Fourier inversions yields

$$I(x, y, t) = \int_0^\infty \kappa \cos(\kappa x) L^{-1} \left\{ \frac{\exp[-(\kappa^2 + s/c)^{1/2} y]}{s(\kappa^2 + s/c)^{1/2}} \right\} d\kappa \quad (A1)$$

Formulae (29.2.14) and (29.3.84) of Abramowitz and Stegun (1964) (hereafter abbreviated AS) yield the following result

$$L^{-1} \left\{ \frac{\exp[-(\kappa^2 + s/c)^{1/2} y]}{(\kappa^2 + s/c)^{1/2}} \right\} = (c/\pi t)^{1/2} \exp[-\kappa^2 ct - y^2/4ct] \quad (A2)$$

Formula (29.2.6) of AS can then be used to express the Laplace transform in equation (A1) as the following integral:

$$L^{-1} \left\{ \frac{\exp[-(\kappa^2 + s/c)^{1/2} y]}{s(\kappa^2 + s/c)^{1/2}} \right\} = \int_0^t (c/\pi\lambda)^{1/2} \exp[-\kappa^2 c\lambda - y^2/4c\lambda] d\lambda \quad (A3)$$

The integration can be accomplished by the change of variable $\lambda = \beta^2$ and the use of AS (7.4.33). The result is

$$L^{-1} \left\{ \frac{\exp[-(\kappa^2 + s/c)^{1/2} y]}{s(\kappa^2 + s/c)^{1/2}} \right\} = (2\kappa)^{-1} \exp(-\kappa y) \{ 1 + \operatorname{erf}[\kappa(ct)^{1/2} - y/(4ct)^{1/2}] - e^{\kappa y} \operatorname{erfc}[\kappa(ct)^{1/2} + y/(4ct)^{1/2}] \} \quad (A4)$$

Substituting into equation (A1), writing $\cos(\kappa x)$ in exponential form, changing variables, and using AS (7.4.36) then yields the final result, given by equation (45).

The inversion of $I^\#$ is lengthier, but proceeds along the same lines. Again interchange the order of the inversions. The Laplace transform can be inverted by using equation (A3) with $y = 0$ and (29.2.6) of AS. The result is

$$L^{-1} \left\{ (c/s^2)/(\kappa^2 + s/c)^{1/2} \right\} = [c\kappa^{-1} - (2\kappa^3)^{-1}] \operatorname{erf}[\kappa(ct)^{1/2}] + \kappa^{-2} (ct/\pi)^{1/2} \exp(-\kappa^2 ct) \quad (A5)$$

Now, $I^\#$ can be written as follows:

$$I^\#(x, y, t) = - \left\{ \left(2ct \frac{\partial^2}{\partial x^2} + 1 \right) \int_0^\infty \exp(\iota \kappa z) \operatorname{erf}[\kappa(ct)^{1/2}] d\kappa + \iota (4ct/\pi)^{1/2} \frac{\partial}{\partial x} \int_0^\infty \exp(-\kappa^2 ct + \iota \kappa z) d\kappa \right\} \quad (A6)$$

where $z = x + \iota y$. The remaining integrals can be done using (7.4.17) and (7.4.2) of AS:

$$\int_0^\infty \exp(-\kappa^2 t + \iota \kappa z) d\kappa = (\pi/4ct)^{1/2} \exp(-z^2/4ct) \operatorname{erfc}[-\iota z/(4ct)^{1/2}] \quad (A7)$$

$$\int_0^\infty \exp(\iota \kappa z) \operatorname{erf}[\kappa(ct)^{1/2}] d\kappa = \iota z^{-1} \exp(-z^2/4ct) \operatorname{erfc}[-\iota z/(4ct)^{1/2}] \quad (A8)$$

The final expression for $I^\#$ is given by equation (51).

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