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BOUNDARY LAYER ANALYSIS OF PLANE STRAIN SHEAR CRACKS PROPAGATING STEADILY ON AN IMPERMEABLE PLANE IN AN ELASTIC DIFFUSIVE SOLID

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ABSTRACT

MATCHED asymptotic expansions are used to examine the stress and pore pressure fields near the tip of a plane strain shear (mode II) crack propagating on an impermeable plane in a linear elastic diffusive solid. For propagation speeds V that are large compared with c/l, where c is the diffusivity and l is the crack-length, boundary layers are required at the crack-tip and on the line ahead of the crack. The latter is required to meet the condition of no flow across this plane; in contrast, for propagation on a permeable plane, a boundary layer is required on the crack faces behind the tip. As for the permeable plane, the solution in the crack-tip boundary layer reveals that the stress field near the crack-tip has the form of the usual linear elastic field with a stress intensity factor $[(1-\nu_u)/(1-\nu)]K^{(e)}$, where $K^{(e)}$ is the stress intensity factor of the outer elastic field, ν is the Poisson's ratio governing slow (drained) deformation, and $\nu_u \geqslant \nu$ is the Poisson's ratio governing rapid (undrained) deformation. Thus, coupling between deformation and fluid diffusion reduces the local value of the stress intensity factor and, hence, stabilizes against rapid propagation. For the permeable plane, the pore pressure goes to zero as the crack-tip is approached along any ray. In contrast, for the impermeable plane, a closed-form solution for the pore pressure in the crack-tip boundary layer reveals that the pore pressure at the crack-tip is non-zero, but bounded.

1. INTRODUCTION

IN AN ordinary linear elastic solid the stress field near the edge of a crack is well known to be singular as the inverse square root of the distance from the crack-tip and to have a universal spatial dependence. This singular field can be used as an outer boundary condition for more detailed investigations of inelastic behavior near the crack-tip and as an inner boundary condition for numerical analysis of cracks in finite linear elastic bodies. The near-tip field in a linear elastic diffusive solid can have similar usefulness. But, for the diffusive solid, the crack-tip field is velocity dependent.

This paper analyses the stress and pore pressure fields near the tip of a shear crack propagating steadily on an impermeable plane in a linear elastic diffusive solid. The analysis complements that of SIMONS (1977) for a shear crack on a permeable plane. As for the permeable plane examined by SIMONS (1977), we will find that for slow propagation the near-tip stress field is identical in form to that for a linear elastic solid. For very rapid (but still quasi-static) propagation, the near-tip field again approaches that for a linear elastic solid. But this solution for rapid propagation is not uniformly valid; in particular, it is not valid at the crack-tip and on the line ahead of the crack. Consequently, it is necessary to introduce boundary layers to determine the field in these regions.

The motivation for this analysis, like that of SIMONS (1977), arises from application to faulting in the earth's crust. RICE and SIMONS (1976), by analysing the propagation of a steadily moving semi-infinite shear crack in a linear elastic diffusive solid, have shown that the coupling between deformation and pore–fluid diffusion can stabilize the fault against rapid propagation for a range of velocities that is consistent with that observed for episodic creep events on the San Andreas fault in California. In addition, RUINA (1978) has examined propagation of a semi-infinite tensile crack in a diffusive solid with applications to hydraulic fracture and CLEARY (1978) has used numerical analysis to study shear and tensile fractures.

The analysis of RICE and SIMONS (1976), as well as the cases of shear cracks studied by CLEARY (1978), are appropriate if the fault is permeable. But, many faults are thought to be impermeable because they contain finely comminuted material (WU *et al.*, 1975; WANG and LIN, 1978) or because they are barriers to fluid flow (LIPPINCOTT *et al.*, 1985). Consequently, RUDNICKI and KOUTSIBELAS (1990) have reexamined the problem considered by RICE and SIMONS (1976) for the impermeable fault. They find similar stabilizing effects due to coupling between deformation and diffusion, but, in addition, find that the change in pore fluid pressure on the fault plane, an effect that is absent for the permeable fault, can affect propagation.

The nature of the pore pressure change near the crack-tip is of particular interest because this change affects the resistance to slip. The resistance to slip is increased not only by an increase in compressive normal stress but also by a decrease in pressure. More specifically, the resistance to slip depends on the effective compressive stress, the total stress minus the pore fluid pressure. Consequently, knowledge of the near-tip behavior of the pore pressure change is needed for formulation of more elaborate models of the slipping process.

2. FORMULATION

The equations governing the response of a linear elastic diffusive solid were first formulated by BIOT (1941) within the context of fluid-saturated soils, but the presentation here follows that of RICE and CLEARY (1976). Their approach emphasizes that the response of the diffusive solid in the limits of deformation that is slow and rapid by comparison to the time scale of fluid diffusion is identical to that in an ordinary elastic solid but with different Poisson's ratios. For slow deformation, said to be drained, the response is that of an elastic solid with Poisson's ratio ν; for rapid, undrained deformation, the Poisson's ratio is ν_u. Because $\nu \leqslant \nu_u \leqslant 1/2$, where the upper limit is attained for separately incompressible constituents and the lower for a highly compressible pore fluid, the undrained response is stiffer than the drained.

For arbitrary plane strain deformations the strain components depend on the stress components $\sigma_{\alpha\beta}$ and pore fluid pressure p as follows (RICE and CLEARY, 1976):

$$2G\varepsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu(\sigma_{xx} + \sigma_{yy})\delta_{\alpha\beta} + 2\eta(1-\nu)p\delta_{\alpha\beta}, \tag{1}$$

where G is the shear modulus, η is a material constant, $\delta_{\alpha\beta}$ is the Kronecker delta and $(\alpha, \beta) = (1, 2) = (x, y)$. An additional constitutive equation is needed for the alteration of fluid mass content per unit volume of porous solid, m, from its reference value m_0:

$$m - m_0 = \eta\rho_0(1-\nu)[(\sigma_{xx}+\sigma_{yy})+(2\eta/\mu)p]/G, \tag{2}$$

where ρ_0 is the mass density of the pore fluid and $\mu = (\nu_u - \nu)/(1-\nu)$. For slow drained deformation, there is no change in the pore pressure and (1) reduces to the usual linear elastic expression. For rapid, undrained deformation $m = m_0$. Solving (2) for p and substituting into (1) again yields the linear elastic expression with ν replaced by ν_u. The final constitutive equation is Darcy's law (RICE and CLEARY, 1976):

$$q_\alpha = -\rho_0\kappa\,\partial p/\partial X_\alpha, \tag{3}$$

where q_α is the mass flow rate in the α direction per unit area and κ is a permeability. Here, coordinates X_α are employed and, immediately below, $X_1 = X$, $X_2 = Y$.

For plane strain deformation, RICE and CLEARY (1976) show that the governing equations can be written entirely in terms of the stress components and the pore fluid pressure. These equations, for deformation in the XY plane, are as follows:

$$\partial\sigma_{xx}/\partial X + \partial\sigma_{yx}/\partial Y = 0, \tag{4}$$

$$\partial\sigma_{xy}/\partial X + \partial\sigma_{yy}/\partial Y = 0, \tag{5}$$

$$\nabla^2(\sigma_{xx}+\sigma_{yy}+2\eta p) = 0, \tag{6}$$

$$(c\nabla^2 - \partial/\partial t)[\sigma_{xx}+\sigma_{yy}+(2\eta/\mu)p] = 0, \tag{7}$$

where c is a diffusivity that can be expressed in terms of parameters already introduced as

$$c = G\kappa\mu/[2\eta^2(1-\nu_u)]$$

and $\nabla^2(\ldots) = \partial^2(\ldots)/\partial X^2 + \partial^2(\ldots)/\partial Y^2$. Equations (4) and (5) are the usual equations of equilibrium in the absence of body forces. Equation (6) arises from compatibility after the constitutive relations have been used to eliminate the strain components. The diffusion equation (7) results from substituting Darcy's law into an equation for conservation of fluid mass and use of the constitutive relation.

The governing equations (4)–(7) will be solved for the case of a semi-infinite shear (mode II) crack located on the X-axis and moving steadily and quasi-statically in the positive X-direction at a constant speed V. The crack is loaded by shear stresses $\tau(X)$ which are applied to the crack faces and move with the crack. The crack and the applied loads are assumed to have been moving at this speed long enough so that transient effects have died out. Thus, the problem is one of steady propagation and any explicit dependence on time t can be eliminated by adopting a coordinate system that moves with the crack-tip. Furthermore, $\partial/\partial t$ can be replaced by $-V\,\partial/\partial X$ and, thus, (7) becomes

$$(c\nabla^2 + V\,\partial/\partial X)[\sigma_{xx}+\sigma_{yy}+(2\eta/\mu)p] = 0. \tag{8}$$

Because of anti-symmetry about the crack plane $Y = 0$, it is possible to formulate the problem in the upper half plane with boundary conditions given on $Y = 0$. The stress component σ_{yy} is anti-symmetric about $Y = 0$ and, because the normal traction is continuous across $Y = 0$, it must be zero there:

$$\sigma_{yy}(X, 0) = 0, \quad -\infty < X < \infty. \tag{9}$$

Similarly, the displacement component u_x is anti-symmetric about $Y = 0$. Although it is discontinuous on the crack itself, it must be continuous ahead of the crack and, hence, must vanish there:

$$u_x(X, 0) = 0, \quad 0 \leqslant X < \infty. \tag{10}$$

This condition can be rewritten in terms of the stresses and pore pressure by using the strain–displacement relation $\varepsilon_{xx} = \partial u_x/\partial X$ and the constitutive relation (1). The result is

$$\sigma_{xx}(X, 0) + 2\eta p(X, 0) = 0, \quad 0 \leqslant X < \infty, \tag{11}$$

where (9) has also been used.

The pore fluid pressure p is also anti-symmetric about $Y = 0$. Hence, if it is continuous on $Y = 0$, it must vanish there and (11) reduces to $\sigma_{xx}(X, 0) = 0$ $(0 \leqslant X < \infty)$. In this case, $\partial p/\partial Y$, in general, will not be zero on $Y = 0$ and, hence, by Darcy's law (3), fluid will flow across the X-axis. Thus, the boundary condition $p = 0$, used in studies of shear cracks in diffusive solids by RICE and SIMONS (1976), SIMONS (1977) and CLEARY (1978), is appropriate for a permeable fault. RUDNICKI (1987) has noted that the crack plane models a shear fault that is impermeable to fluid flow if

$$\partial p(X, 0)/\partial Y = 0, \quad -\infty < X < \infty. \tag{12}$$

The pore fluid pressure is still required to be anti-symmetric about $Y = 0$, but need not be continuous and, hence, takes on equal and opposite values as the X-axis is approached from above or below.

As already mentioned, the applied shear loading on the fault faces is described by $\tau(X)$. This provides the following boundary condition:

$$\sigma_{xy}(X, 0) = -\tau(X), \quad -\infty < X \leqslant 0. \tag{13}$$

As discussed by RICE and SIMONS (1976), $\tau(X)$ represents the difference between an applied farfield loading and a resistive friction stress.

Finally, we follow SIMONS (1977) and specify the following sufficient conditions for uniqueness of solution:

$$\sigma_{\alpha\beta}(R, \theta), \quad |\nabla p| = O(R^{-1/2}) \quad \text{as } R \to 0, \tag{14}$$

$$\sigma_{\alpha\beta}(R, \theta), \quad |\nabla p| = O(R^{-3/2}) \quad \text{as } R \to \infty, \tag{15}$$

uniformly in $-\pi < \theta < \pi$ where $\theta = \arctan\,(Y/X)$, $R = (X^2 + Y^2)^{1/2}$, and $|\nabla p| = [(\partial p/\partial X)^2 + (\partial p/\partial Y)^2]^{1/2}$. As noted by SIMONS (1977), these reduce to the usual conditions of bounded strain energy when $V = 0$.

To facilitate the asymptotic analysis to come, we introduce non-dimensional spatial variables $(x, y) = (X/l, Y/l)$, where l is some characteristic length of the load distribution. With this change of variables, the field equations (4)–(6) and boundary conditions (9)–(13) retain the same form (with (X, Y) replaced by (x, y)), but (8) becomes

$$(\varepsilon^2\nabla^2+\partial/\partial x)[\sigma_{xx}+\sigma_{yy}+(2\eta/\mu)p] = 0, \tag{16}$$

where $\varepsilon^2 = c/Vl$ and, now, $\nabla^2(\ldots) = \partial^2(\ldots)/\partial x^2+\partial^2(\ldots)/\partial y^2$. Note that ε^2 is the ratio of a characteristic time of loading l/V to a characteristic time for diffusion l^2/c.

3. Slow Crack Propagation

As discussed in connection with the constitutive relations, for deformations that are slow by comparison to the time-scale of fluid mass diffusion, i.e. drained conditions, the alteration in the pore fluid pressure p will be zero and the response of a diffusive solid reduces to that of an ordinary linear elastic solid with shear modulus G and Poisson's ratio ν. Thus, it is anticipated that the solution for the crack propagation speed V approaching zero will be identical to that of an ordinary linear elastic solid. This supposition can be verified by taking the limit $\varepsilon \to \infty$ in (16) which is the only equation involving ε. Dividing through by ε^2 and setting $\varepsilon \to \infty$ yields

$$\nabla^2[\sigma_{xx}^{(s)}+\sigma_{yy}^{(s)}+(2\eta/\mu)p^{(s)}] = 0, \tag{17}$$

where the superscript (s) denotes the solution for slow propagation, $\varepsilon \to \infty$. Combining (17) with (6) reveals that $p^{(s)}$ satisfies Laplace's equation. The solution satisfying the boundary condition (12) and the order conditions (14, 15) is $p^{(s)} \equiv 0$. Substituting this result into (6) and (17) reduces both equations to the usual compatibility condition for plane elasticity in the absence of body forces. Thus, in the limit $V \to 0$, the solution for the stresses $\sigma_{\alpha\beta}^{(s)}$ is identical to that for the plane elasticity problem and to those found by Simons (1977) for the permeable shear crack. Obviously, this results because the identically zero pore pressure field, appropriate for drained response, satisfies the boundary condition for both a permeable crack and an impermeable crack.

The solution for the stresses near the edge of a crack in a linear elastic body is well known to have the following form (e.g. Rice, 1968):

$$\sigma_{\alpha\beta}^{(e)} = K^{(e)}(2\pi r)^{-1/2}f_{\alpha\beta}(\theta)+O(1) \tag{18}$$

as $r \to 0$, where the superscript (e) denotes the elastic solution and $K^{(e)}$ is the mode II stress intensity factor which depends on distribution of loading $\tau(x)$. The functions $f_{\alpha\beta}$, which are identical for all mode II crack problems, are as follows:

$$\begin{aligned} f_{xx}(\theta) &= -\sin(\theta/2)[2+\cos(\theta/2)\cos(3\theta/2)], \\ f_{xy}(\theta) &= \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)], \\ f_{yy}(\theta) &= \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2). \end{aligned} \tag{19}$$

4. RAPID CRACK PROPAGATION

By arguments analogous to those used for slow crack propagation, one might anticipate that undrained behavior pertains in the limit $V \to \infty$. As noted by SIMONS (1977) for the permeable crack, this turns out to be true, but not uniformly so. More specifically, the response near the crack tip and, for the impermeable crack, on the line ahead of the crack ($y = 0$, $x > 0$) is not undrained. In contrast, the elastic solution obtained for slow crack propagation applies everywhere in the (x, y) plane and, in particular, the solution given by (18) and (19) gives the correct asymptotic behavior near the crack-tip.

4.1. *Outer expansion*

The limit of rapid crack propagation corresponds to taking the limit $\varepsilon \to 0$ in (16):

$$\frac{\partial}{\partial x}[\sigma_{xx}^{(r)} + \sigma_{yy}^{(r)} + (2\eta/\mu)p^{(r)}] = 0, \tag{20}$$

where the superscript (r) denotes the solution for rapid propagation. Note that, in contrast to the limit $\varepsilon \to \infty$ for slow propagation, the limiting process for rapid propagation reduces the order of the equation. This is a signal that the perturbation is singular and that boundary layers will be needed to meet the boundary condition (KEVORKIAN and COLE, 1981). The other equations (4)–(6) retain the same form in this limit. Equation (20) requires that the quantity [...] be a function only of y. But, because this quantity is proportional to the fluid mass content, it must vanish as $x \to \infty$ and, hence, must be identically zero. Consequently, the pore pressure $p^{(r)}$ is related to the mean stress as for undrained response:

$$p^{(r)} = -(\mu/2\eta)[\sigma_{xx}^{(r)} + \sigma_{yy}^{(r)}]. \tag{21}$$

Substituting (21) into (6) reveals that the mean stress satisfies Laplace's equation:

$$\nabla^2(\sigma_{xx}^{(r)} + \sigma_{yy}^{(r)}) = 0. \tag{22}$$

Thus, the stress components $\sigma_{\alpha\beta}^{(r)}$ satisfy (22), which expresses compatibility for plane elasticity, and the two equilibrium equations (4), (5), which retain the same form in the limit $\varepsilon \to 0$. Consequently, the zeroth order outer solution for the stresses is again given by the elasticity solution, i.e.

$$\sigma_{\alpha\beta}^{(r)} \equiv \sigma_{\alpha\beta}^{(e)} \tag{23}$$

and the pore pressure is given by (21) as $\varepsilon \to 0$.

Now, consider the boundary conditions (9), (11)–(13), which also retain the same form as $\varepsilon \to 0$. Because of the reduction of order which occurs in taking the limit $\varepsilon \to 0$, it is anticipated that the zeroth order solution will not meet all the boundary conditions. In particular, it is easily shown that the condition (12) is violated as $r \to 0$. Consider the expression for $\partial p^{(r)}/\partial y$ near the crack-tip, obtained by using (23) and substituting (18) and (19) into (21):

$$\frac{\partial p^{(r)}}{\partial y}(x, y) = (\mu/2\eta)\frac{K^{(c)}}{(2\pi r^3)^{1/2}}\cos(3\theta/2) \tag{24}$$

as $r \to 0$. This expression vanishes for $\theta = \pi$ and, hence, meets the boundary condition (12) for $x < 0$, but not for $x > 0$. Furthermore, (24) violates the order condition (14) as $r \to 0$. This suggests that inner expansions are needed as $r \to 0$ and as $y \to 0$ for $x > 0$.

Note that the pore pressure itself, which is given by

$$p^{(r)}(x, y) = (\mu/\eta)\frac{K^{(c)}}{(2\pi r)^{1/2}}\sin(\theta/2) \tag{25}$$

as $r \to 0$, satisfies the condition for a permeable shear crack ($p(x, y = 0) = 0$) on $\theta = 0$ ($x > 0$), but not on $\theta = \pi$ ($x < 0$). Thus, a crack face boundary layer is required for $x < 0$ in the analysis of the permeable shear crack by SIMONS (1977) whereas a crack-line boundary layer is required for $x > 0$ in the present case.

4.2. *Crack-tip inner expansion*

The discussion of the preceding section indicates that a crack-tip expansion is needed as $r \to 0$. To effect this expansion, we follow SIMONS (1977) and introduce crack-tip coordinates $\hat{x}$, $\hat{y}$ given by

$$\hat{x} = x/\varepsilon^2, \quad \hat{y} = y/\varepsilon^2. \tag{26}$$

This choice is motivated by a desire to retain the $\nabla^2(\ldots)$ operator in (16) under the crack-tip limiting process $\varepsilon \to 0$ with (x, y) fixed. With this change of variables, the field equations (4)–(6), (16), retain the same form.

The appropriate form for the expansion of the crack-tip solution, denoted by $\hat{\sigma}_{\alpha\beta}$, $\hat{p}$ is dictated by the expectation that the inner and outer expansions have a domain of overlap (e.g. KEVORKIAN and COLE, 1981). More specifically, this expectation is fulfilled by the requirement that the one-term outer expansion (limit $\varepsilon \to 0$, with (x, y) fixed) of the crack-tip inner solution equal the one-term inner expansion (limit $\varepsilon \to 0$, with $(\hat{x}, \hat{y})$ fixed) of the outer solution. This statement is a special case of the matching principle stated in greater generality by VAN DYKE (1975). The inner expansion of the outer solution is obtained by rewriting the outer solution in terms of the crack-tip variables $\hat{x}$, $\hat{y}$ and expanding as $\varepsilon \to 0$ with $(\hat{x}, \hat{y})$ fixed. For example, as $r = \varepsilon^2\hat{r} \to 0$, the leading term of the outer solution for the pore pressure $p^{(r)}$ is given by (25). When rewritten in terms of $\hat{r}$, this equation becomes

$$p^{(r)}(\varepsilon^2\hat{x}, \varepsilon^2\hat{y}) = (\mu/\eta)\frac{K^{(c)}}{\varepsilon(2\pi\hat{r})^{1/2}}\sin(\theta/2) \tag{27}$$

as $\varepsilon \to 0$. Therefore, the matching principle requires the crack-tip inner solution for the pore pressure to have the following form:

$$\hat{p}(\hat{x}, \hat{y}; \varepsilon) = \varepsilon^{-1}\hat{p}^{(1)}(\hat{x}, \hat{y}) + o(\varepsilon^{-1}) \tag{28}$$

as $\varepsilon \to 0$. Similarly, the expansions for the stress components have the form

$$\hat{\sigma}_{\alpha\beta}(\hat{x}, \hat{y}; \varepsilon) = \varepsilon^{-1}\hat{\sigma}^{(1)}_{\alpha\beta}(\hat{x}, \hat{y}) + o(\varepsilon^{-1}) \tag{29}$$

as $\varepsilon \to 0$. Substituting into the governing equations (4)–(6), (16) and taking the limit $\varepsilon \to 0$ yields the following equations:

$$\partial\hat{\sigma}^{(1)}_{xx}/\partial\hat{x} + \partial\hat{\sigma}^{(1)}_{yx}/\partial\hat{y} = 0, \quad \partial\hat{\sigma}^{(1)}_{xy}/\partial\hat{x} + \partial\hat{\sigma}^{(1)}_{yy}/\partial\hat{y} = 0, \tag{30}$$

$$(\partial^2/\partial\hat{x}^2 + \partial^2/\partial\hat{y}^2)[\hat{\sigma}^{(1)}_{xx} + \hat{\sigma}^{(1)}_{yy} + 2\eta\hat{p}^{(1)}] = 0, \tag{31}$$

$$(\partial^2/\partial\hat{x}^2 + \partial^2/\partial\hat{y}^2 + \partial/\partial\hat{x})[\hat{\sigma}^{(1)}_{xx} + \hat{\sigma}^{(1)}_{yy} + (2\eta/\mu)\hat{p}^{(1)}] = 0. \tag{32}$$

Application of the same operations to boundary conditions (9), (11)–(13) yields

$$\hat{\sigma}^{(1)}_{yy}(\hat{x}, 0) = 0, \quad \partial\hat{p}^{(1)}(\hat{x}, 0)/\partial\hat{y} = 0, \quad -\infty < \hat{x} < \infty, \tag{33}$$

$$\hat{\sigma}^{(1)}_{xx}(\hat{x}, 0) + 2\eta\hat{p}^{(1)}(\hat{x}, 0) = 0, \quad 0 \leqslant \hat{x} < \infty, \tag{34}$$

$$\hat{\sigma}^{(1)}_{xy}(\hat{x}, 0) = 0, \quad -\infty < \hat{x} \leqslant 0. \tag{35}$$

The first of the order conditions (14) is replaced by

$$\hat{\sigma}^{(1)}_{\alpha\beta} = K^{(e)}(2\pi\hat{r})^{-1/2}f_{\alpha\beta}(\theta) + o(r^{-1/2}) \tag{36}$$

as $\hat{r} \to \infty$. This is a restatement of the requirement that the crack-tip solution match-up with the outer solution and is a mathematical formulation of the condition of "small-scale yielding" in fracture mechanics (RICE, 1968; EDMUNDS and WILLIS, 1976).

Fourier transforms and the Wiener–Hopf technique are used to solve (30)–(32) subject to (33)–(36) as outlined in Appendix A. The results for the mean stress and pore pressure can be obtained in closed form and are as follows:

$$\tfrac{1}{2}(\hat{\sigma}^{(1)}_{xx} + \hat{\sigma}^{(1)}_{yy}) = \frac{-K^{(e)}}{(2\pi\hat{r})^{1/2}}\{\sin(\theta/2)\{1 - \mu\, e^{\hat{r}\sin^2(\theta/2)}\operatorname{erfc}[\hat{r}^{1/2}\sin(\theta/2)]\}$$
$$+ \mu\cos(\theta/2)D[\hat{r}^{1/2}\cos(\theta/2)]\}, \tag{37}$$

$$\eta\hat{p}^{(1)} = \frac{\mu K^{(e)}}{(2\pi\hat{r})^{1/2}}\{\sin(\theta/2)\{1 - e^{\hat{r}\sin^2(\theta/2)}\operatorname{erfc}[\hat{r}^{1/2}\sin(\theta/2)]\}$$
$$+ \cos(\theta/2)D[\hat{r}^{1/2}\cos(\theta/2)]\}, \tag{38}$$

where erfc (z) is the complementary error function (ABRAMOWITZ and STEGUN, 1964, eq. (7.1.2)) and $D(z)$ is $(2/\pi^{1/2})$ times Dawson's integral (ABRAMOWITZ and STEGUN, 1964). The expressions for the shear stress and stress difference involve an integral that cannot be done in finite form:

$$\hat{\sigma}^{(1)}_{xy} = \frac{K^{(e)}}{(2\pi\hat{r})^{1/2}}\cos(\theta/2)[1 - \sin(\theta/2)\sin(3\theta/2)] + \mu(K^{(e)}/2^{1/2}\pi)\, e^{\pi i/4}\left\{\frac{\partial H}{\partial y} - i\frac{\partial K}{\partial x}\right\}, \tag{39}$$

$$\tfrac{1}{2}(\hat{\sigma}^{(1)}_{yy} - \hat{\sigma}^{(1)}_{xx}) = \frac{K^{(e)}}{(2\pi\hat{r})^{1/2}}\sin(\theta/2)[1 + \cos(\theta/2)\cos(3\theta/2)]$$
$$+ \mu(K^{(e)}/2^{1/2}\pi)\, e^{i\pi/4}\left\{2H + \frac{\partial H}{\partial x} + i\frac{\partial K}{\partial y}\right\}, \tag{40}$$

where

$$H(x, y) = 2(\pi/r)^{1/2}\, e^{-\pi i/4}\, \{\sin(\theta/2)\, e^{r\sin^2(\theta/2)}\, \mathrm{erfc}\,[\sqrt{r}\sin(\theta/2)] - \cos(\theta/2) D[\sqrt{r}\cos(\theta/2)]\}, \quad (41)$$

$$K(x, y) = (2/r)^{1/2}\, e^{\pi i/4} \int_0^{\infty} e^{-s} \frac{\{\sin(\theta+\phi/2)+\cos(\theta+\phi/2)\}}{\{s^2-2sr\cos(\theta)+r^2\}^{1/4}}\, ds \quad (42)$$

and the angle $\phi(s)$ is defined by

$$\tan\phi = (s\cos(\theta)-r)/\sin(\theta).$$

Although these expressions are lengthy, it can be shown that in the limit $\hat{r}\to 0$ the stress components approach the form of the elastic solution (18) with the stress intensity factor $K^{(e)}$ replaced by $(1-\mu)K^{(e)}$. Thus the stress intensity at the crack-tip is reduced by a factor of $(1-\mu)$. This result is identical to that obtained for the permeable shear crack (RICE and SIMONS, 1976; SIMONS, 1977) and, as discussed by RICE and SIMONS (1976), is due to the constraint of the stiffer, undrained response of the material surrounding the crack-tip region.

The pore pressure at the crack-tip approaches the finite limit

$$\eta\hat{p}^{(1)} = 2^{1/2}\mu K^{(e)}/\pi + O(\hat{r}^{1/2}) \quad (43)$$

as $\hat{r}\to 0$ in $0 \leqslant \theta \leqslant \pi$. (Recall that the pore pressure is discontinuous on $y = 0$ and, hence, approaches a value equal in magnitude but opposite in sign to (43) for $-\pi \leqslant \theta \leqslant 0$.) In contrast, SIMONS (1977) has shown that for the permeable shear crack, the pore pressure goes to zero as the crack-tip is approached along any ray. Although the pore pressure is not zero at the crack-tip for the impermeable shear crack, the crack-tip region can be considered drained because the pore pressure is bounded and, hence, the ratio of the pore pressure to any stress component goes to zero as the crack-tip is approached (except, of course, for special rays along which certain stress components vanish.)

4.3. *Crack-line inner expansion*

The crack-line inner expansion is needed because of the failure of the outer (undrained) solution to meet the boundary condition (12) on $y = 0$, $0 \leqslant x < \infty$. (In contrast, a crack-face expansion on $y = 0$, $x \leqslant 0$ is needed for the permeable shear crack.) Thus, the crack-line expansion is expected to describe one-dimensional diffusion that effects the transition from $\partial p/\partial y = 0$ on $y = 0$ to the expression (24) given by the outer solution. As in the case of the similar analysis for the permeable crack (SIMONS, 1977), the appropriate choice of the scaling is dictated by a desire to reduce (8) to an equation of one-dimensional diffusion in the y direction:

$$\tilde{x} = x \quad \tilde{y} = y/\varepsilon. \quad (44)$$

The matching principle requires that the one-term outer expansion ($\varepsilon\to 0$ with x, y fixed) of the crack-line solution equal the inner expansion ($\varepsilon\to 0$ with $\tilde{x}$, $\tilde{y}$ fixed) of the outer solution. This suggests the following form for the expansion of the crack-line solution:

$$\tilde{\sigma}_{xy}(\tilde{x}, \tilde{y}; \varepsilon) = \tilde{\sigma}_{xy}^{(1)}(\tilde{x}, \tilde{y}) + o(1) \tag{45}$$

$$[\tilde{\sigma}_{yy}, \tilde{\sigma}_{xx}, \tilde{p}](\tilde{x}, \tilde{y}; \varepsilon) = \varepsilon[\tilde{\sigma}_{yy}^{(1)}, \tilde{\sigma}_{xx}^{(1)}, \tilde{p}^{(1)}](\tilde{x}, \tilde{y}) + o(\varepsilon). \tag{46}$$

In contrast to the crack-face solution for the permeable shear crack, the expansion for $\tilde{p}$ begins at order ε. Substitution of the scaling (44) and the expansions (45), (46) into the field equations (4)–(6), (16), and taking the limit $\varepsilon \to 0$ yields the following equations governing the first term of the crack-line solution:

$$\partial\tilde{\sigma}_{xy}^{(1)}/\partial\tilde{y} = 0, \quad \partial\tilde{\sigma}_{xy}^{(1)}/\partial\tilde{x} + \partial\tilde{\sigma}_{yy}^{(1)}/\partial\tilde{y} = 0, \tag{47}$$

$$\partial^2\{\tilde{\sigma}_{xx}^{(1)} + \tilde{\sigma}_{yy}^{(1)} + 2\eta\tilde{p}^{(1)}\}/\partial\tilde{y}^2 = 0, \tag{48}$$

$$(\partial^2/\partial\tilde{y}^2 + \partial/\partial\tilde{x})\{\tilde{\sigma}_{xx}^{(1)} + \tilde{\sigma}_{yy}^{(1)} + (2\eta/\mu)\tilde{p}^{(1)}\} = 0. \tag{49}$$

Application of the same operations to the boundary conditions yields

$$\partial\tilde{p}^{(1)}/\partial\tilde{y} = 0, \quad \tilde{\sigma}_{yy}^{(1)} = 0, \quad \tilde{\sigma}_{xx}^{(1)} + 2\eta\tilde{p}^{(1)} = 0 \tag{50}$$

for $\tilde{y} = 0, \tilde{x} > 0$. The requirement that the crack-line solution match the outer solution results in the following conditions:

$$\tilde{\sigma}_{xy}^{(1)} \simeq \sigma_{xy}^{(e)}(\tilde{x}, 0), \tag{51}$$

$$\tilde{\sigma}_{xx}^{(1)} \simeq \tilde{y}\frac{\partial\sigma_{xx}^{(e)}}{\partial y}(\tilde{x}, 0), \quad \tilde{\sigma}_{yy}^{(1)} \simeq \tilde{y}\frac{\partial\sigma_{yy}^{(e)}}{\partial y}(\tilde{x}, 0), \tag{52}$$

$$\tilde{p}^{(1)} \simeq -(\mu/2\eta)\tilde{y}\frac{\partial}{\partial y}(\sigma_{xx}^{(e)} + \sigma_{yy}^{(e)})(\tilde{x}, 0) \tag{53}$$

as $\tilde{y} \to \infty$.

The crack-line solution must also be compatible with the crack-tip solution. More specifically, the one-term crack-tip expansion of the crack-line solution must equal the one-term crack-line expansion of the crack-tip solution. The crack-line expansion of the crack-tip solution is obtained by rewriting (37)–(40) in terms of the crack-line variables, i.e.

$$\hat{x} = \tilde{x}/\varepsilon^2, \quad \hat{y} = \tilde{y}/\varepsilon \tag{54}$$

and expanding for small ε. For example, the result of retaining the first term for the pore fluid pressure and rewriting in terms of the crack-tip variables is as follows:

$$(2\eta/\mu)\hat{p} = \frac{K^{(e)}2^{1/2}}{\varepsilon\pi\hat{x}}\{1 + [\hat{y}/(4\hat{x}/\pi)^{1/2}]\{1 - e^{\hat{y}^2/4\hat{x}}\,\mathrm{erfc}\,[\hat{y}/(4\hat{x})^{1/2}]\}\}. \tag{55}$$

However, it turns out that this condition is met automatically and does not need to be enforced.

The solution to the system of equations defining the one-term crack-line solution is obtained in Appendix B. The results are as follows:

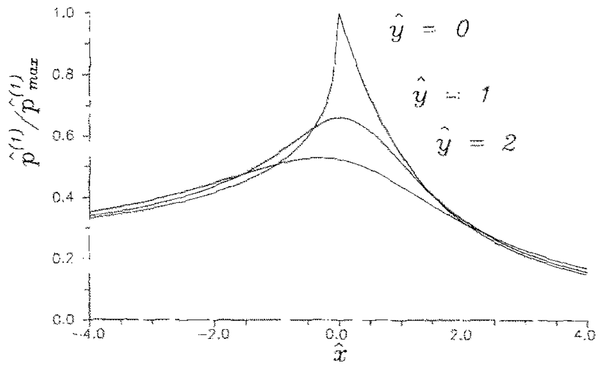

FIG. 1. Pore pressure in the crack-tip boundary layer divided by the value at the crack-tip (see Eq. 43) against $\hat{x} = x/\varepsilon^2$ for three fixed values of $\hat{y} = y/\varepsilon^2$: 0^+, 1, 2. Values for $y \leqslant 0$ are equal in magnitude and opposite in sign to those shown.

$$\tilde{\sigma}_{xy}^{(1)} = \sigma_{xy}^{(e)}(\tilde{x}, 0), \quad \tilde{\sigma}_{yy}^{(1)} = -y\frac{\partial \sigma_{xy}^{(e)}}{\partial x}(\tilde{x}, 0), \tag{56}$$

$$\tilde{\sigma}_{xx}^{(1)} = \tilde{y}\frac{\partial \sigma_{xx}^{(e)}}{\partial y}(\tilde{x}, 0) + 2\mu\int_{\tilde{x}}^{\infty}\frac{\exp[-\tilde{y}^2/4(\xi-\tilde{x})]}{[\pi(\xi-\tilde{x})]^{1/2}}\frac{\partial \sigma^{(e)}}{\partial y}(\xi, 0)\,\mathrm{d}\xi, \tag{57}$$

$$(\eta/\mu)\tilde{p}^{(1)} = -\tilde{y}\frac{\partial \sigma^{(e)}}{\partial y}(\tilde{x}, 0) - \int_{\tilde{x}}^{\infty}\frac{\exp[-\tilde{y}^2/4(\xi-\tilde{x})]}{[\pi(\xi-\tilde{x})]^{1/2}}\frac{\partial \sigma^{(e)}}{\partial y}(\xi, 0)\,\mathrm{d}\xi, \tag{58}$$

where $\sigma^{(e)} = (\sigma_{yy}^{(e)} + \sigma_{xx}^{(e)})/2$. Rewriting $\tilde{p}^{(1)}$ in terms of the crack-tip variables, expanding for small ε, and using (46) yields (55).

Analogous results apply for the stress components. Note that the crack-line solution depends on the full outer solution $\sigma_{\alpha\beta}^{(e)}$ and, hence, on the details of the applied loading not just the singular portion as is the case for the crack-tip inner solution.

5. DISCUSSION

The main difference between shear crack propagation on permeable and impermeable planes is that a change in pore pressure is induced on the impermeable plane. For the permeable plane, the pore pressure is required to be zero there by the symmetry of the problem. This difference is most evident in examining the pore pressure in the crack-tip boundary layer (38). The solutions in the crack-tip boundary layer depend on the applied loading only through the stress intensity factor and, hence, are relatively independent of the details of that loading.

Figure 1 plots the pore pressure induced on the impermeable plane $y = 0^+$, divided by its maximum value (see Eq. 43) against $\hat{x}$. (Recall that the pore pressure is discontinuous on $y = 0$ and the value for $y = 0^-$ is equal in magnitude and opposite in sign.) Also shown for comparison is the pore pressure for two nonzero, fixed values of $\hat{y}$, 1.0 and 2.0. As indicated, the maximum pore pressure change occurs at the

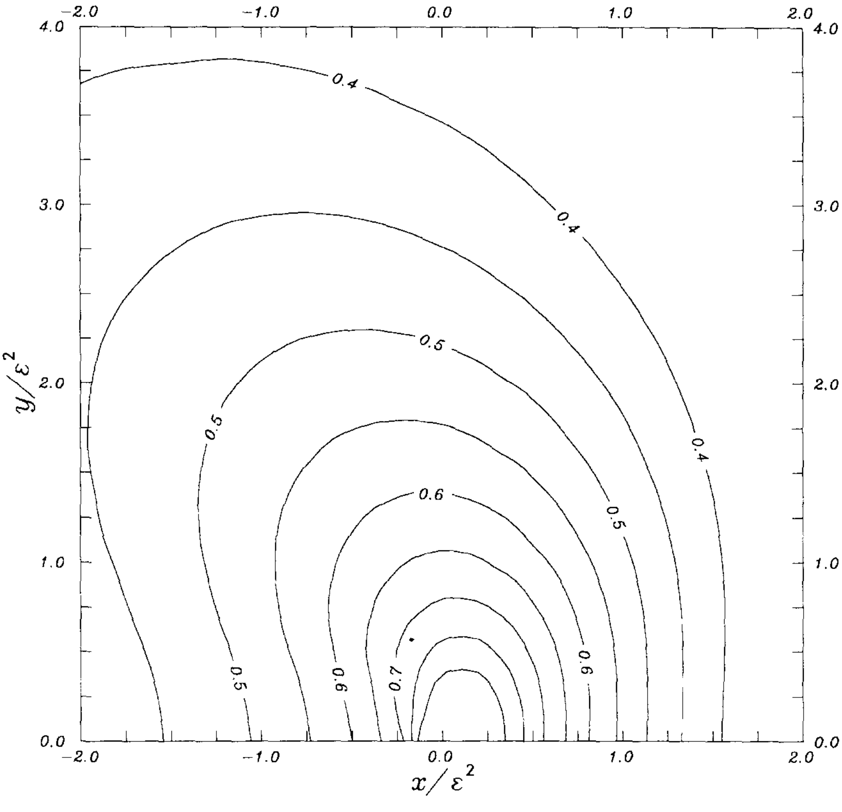

FIG. 2(a). Contours of pore pressure in the crack-tip boundary layer divided by the value at the crack-tip (see Eq. 43). Values for $y \leqslant 0$ are equal in magnitude and opposite in sign to those shown.

crack-tip; it diminishes with distance away from the crack-tip, but more rapidly ahead of the crack-tip than behind.

Figure 2(a) shows contours of the pore pressure change in the crack-tip boundary layer (again, divided by the value at the crack-tip). As required by the boundary condition, the contours intersect $y = 0$ at right angles. The shape of the contours is similar to those for the shear dislocation moving steadily on an impermeable plane (Rudnicki and Roeloffs, 1990), but for the dislocation the pore pressure change becomes unbounded as the dislocation is approached. For comparison, Fig. 2(b) shows the pore pressure contours (divided by the same value as for Fig. 2a) in the crack-tip boundary layer of the shear crack on a permeable plane. The pore pressure change is required to be zero on the crack-line and Simons (1977) has shown that the pore pressure tends to zero as the crack-tip is approached along any ray. Consequently, the magnitude of the pore pressure change is much smaller in the neighbourhood of the crack-tip than for the impermeable plane. In particular, note that the maximum pore pressure occurs near $(\hat{x}, \hat{y}) \simeq (0, 2.5)$ and is much less than the pore pressure change for the impermeable crack at the same location.

Solutions in the crack-line boundary layer, like those in the crack-face boundary

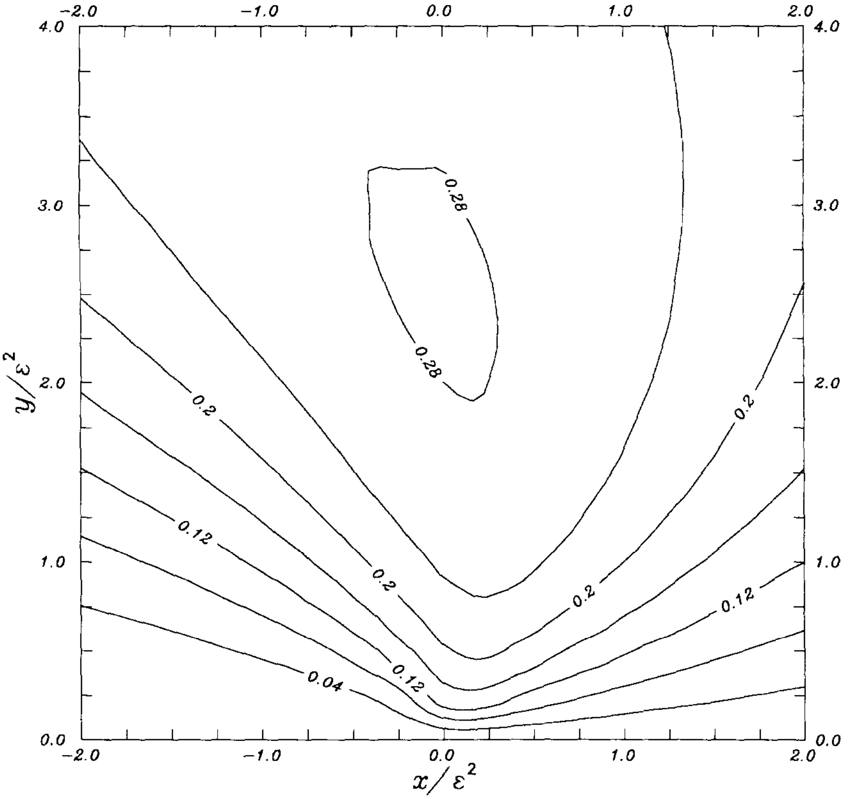

Fig. 2(b) Same as (a) but for the shear crack on a permeable plane. (Pore pressure is divided by the same value as in (a).)

layer in SIMONS' (1977) analysis, depend on the full outer solution. To evaluate these solutions, we assume a uniform shear loading of unit magnitude over a distance l behind the crack-tip. When expressed in nondimensional variables, the boundary condition (13) for this loading becomes

$$\sigma_{xy}(x,0) = -H(x+1), \tag{59}$$

where $H(\ )$ is the Heaviside step function. The derivative of the mean stress, required to evaluate the pore pressure, is easily obtained from results given by TADA *et al.* (1973):

$$\frac{\partial \sigma^{(c)}}{\partial y}(\tilde{x},0) = \frac{-1}{\pi \tilde{x}^{3/2}(1+\tilde{x})}. \tag{60}$$

Substitution of (60) into (58) makes it possible to obtain an expression for the pore pressure in the crack-line boundary layer although all of the integrals cannot be expressed in closed form. Because the crack-line boundary layer accommodates a transition of $\partial p/\partial y$ from zero on the crack-line to a value appropriate for undrained

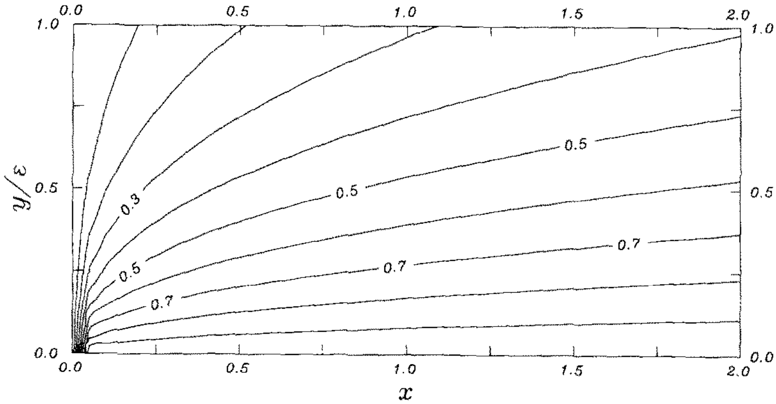

FIG. 3. Contours of the crack-line drainage parameter defined by (61).

response, it is, however, more revealing to examine a drainage parameter, analogous to that introduced by SIMONS (1977). In contrast to that for the permeable crack, the drainage parameter here is defined in terms of the derivative of the pore pressure as follows:

$$\tilde{\Lambda} = \left[\frac{\partial p^{(r)}}{\partial \tilde{y}} - \frac{\partial p}{\partial \tilde{y}}\right] \Big/ \frac{\partial p^{(r)}}{\partial \tilde{y}}, \tag{61}$$

where $p^{(r)}$ is given by (21) and its derivative is to be evaluated at $(\tilde{x}, 0)$. The drainage parameter varies from one on the crack-line to zero where $\partial p/\partial y$ attains the value given by the outer undrained solution.

For the loading (59), the $\partial p^{(r)}/\partial y$ is minus (μ/η) times the right-hand side of (60). After substituting (60) into (58), differentiating, and rearranging the integrals, the drainage parameter can be expressed as follows:

$$\begin{aligned}\tilde{\Lambda} = {} & \exp(\tilde{y}^2/4x)\,\mathrm{erfc}(\tilde{y}/\sqrt{4x}) \\ & - \frac{\tilde{y}}{\sqrt{x}}\left[1 + \frac{1}{(1+x)}\right]\{1 - \sqrt{\pi}(\tilde{y}/\sqrt{4x})\exp(\tilde{y}^2/4x)\,\mathrm{erfc}(\tilde{y}/\sqrt{4x})\} \\ & + \frac{\tilde{y}}{\sqrt{\pi x}}\frac{x^2}{(1+x)}\int_0^\infty \frac{t\exp(-\tilde{y}^2/4x)\,\mathrm{d}t}{(t^2+1)^{3/2}[(1+x)t^2+x]}.\end{aligned} \tag{62}$$

Figure 3 plots contours of $\tilde{\Lambda}$. For large and small values of x $(=\tilde{x})$ and a fixed value of $\gamma = \tilde{y}/(4x)^{1/2}$, $\tilde{\Lambda}$ approaches a constant indicating that the contours are asymptotic to parabolae. The bunching of the contours as x becomes small signals the approach of the crack-tip boundary layer region. Indeed, the crack-line solution becomes inappropriate for values of x of the order of ε^2. As discussed by SIMONS (1977) for the permeable crack, the contours indicate the shape of the boundary layer, but there is no unambiguous way to assign a thickness to the layer.

As mentioned already, the local stress intensity factor in the crack-tip boundary layer region is reduced by a factor of $1-\mu$ $(=(1-\nu_u)/(1-\nu))$ from its value in the limit of very slow propagation. More detailed numerical calculations for specific crack-face loading reveal that for both the permeable (RICE and SIMONS, 1976) and the impermeable (RUDNICKI and KOUTSIBELAS, 1990) planes the stress intensity factor decreases monotonically with velocity between the two limits identified here. The details of the velocity dependence are, however, different for the permeable and impermeable planes. As discussed more thoroughly by RICE and SIMONS (1977) (and by RUDNICKI and KOUTSIBELAS, 1990, for the impermeable crack), the decrease of the stress intensity factor with velocity means that the coupling between deformation and diffusion stabilize the crack against rapid growth. To be more specific, consider that propagation occurs when the energy release rate $\mathscr{G}$ attains a critical value $\mathscr{G}_{\text{crit}}$, where $\mathscr{G}_{\text{crit}}$ is assumed to be a material property. The energy release rate is given in terms of the stress intensity factor by (IRWIN, 1960; RICE, 1968) $\mathscr{G} = K^2/2G(1-\nu)$. Thus, the propagation criteria can be expressed as

$$\mathscr{G}_{\text{nom}}/\mathscr{G}_{\text{crit}} = 1/[K(V)/K(0)]^2 \tag{63}$$

where $\mathscr{G}_{\text{nom}}$ is the value of the energy release rate computed from the stress intensity factor for zero velocity and the velocity dependence of K has been indicated explicitly on the right-hand side. Because $K(V)/K(0)$ is a decreasing function of velocity, the energy supplied by the applied loading, expressed by $\mathscr{G}_{\text{nom}}$, must increase with velocity to maintain $\mathscr{G} = \mathscr{G}_{\text{crit}}$.

The conclusions discussed above pertain for conditions of small-scale yielding, that is, when a model of the crack-tip processes in terms of a singular stress field is adequate. By using a cohesive zone model, RICE and SIMONS (1976) show that at propagation speeds high enough so that the diffusion length c/V approaches the cohesive zone size, the energy release rate decreases with velocity. Thus, there is a maximum velocity for which the porous media effects stabilize propagation. RUDNICKI and KOUTSIBELAS (1990) have obtained results for the impermeable shear crack that are qualitatively similar.

The stabilizing effects just discussed, result from the stiffer response of the porous material surrounding the crack to rapid deformations. Another effect that arises for the impermeable crack is due to the alteration of the frictional shear resistance by the pore pressure induced on the plane of the crack. The shear resistance is proportional to the effective compressive normal stress on the crack, that is, the difference between the total normal compressive stress and the pore pressure. Thus, an increase in pore pressure decreases the frictional resistance and a decrease increases the frictional resistance. Because the values of the pore pressure induced on the impermeable plane as it is approached from above or below are equal in magnitude but opposite in sign, it may not be clear which of these two possibilities pertains. Note, however, that the impermeable plane in this analysis idealizes a finite but narrow width zone of relatively impermeable material. Following a suggestion of RICE (personal communication, 1987), RUDNICKI and KOUTSIBELAS (1990) argue that the crack will tend to follow the path of least resistance within this zone and, as a consequence, it is the pore pressure increase that is most significant in affecting propagation.

The solution for the pore pressure in the crack-tip boundary layer region indicates that the magnitude of the pore pressure change can be substantial. The maximum pore pressure change (43) is proportional to the effective shear loading and to $(Vl/c)^{1/2}$.

(Numerical calculations by RUDNICKI and KOUTSIBELAS, 1990, for a model that takes approximate account of a small cohesive zone at the crack-tip suggest a logarithmic dependence on velocity.) Although the increase in pore pressure decreases the frictional resistance and, hence, is destabilizing, the calculations of RUDNICKI and KOUTSIBELAS (1990) indicate that the amount of destabilization decreases with velocity. Although the maximum pore pressure change increases with velocity, the distribution of the pore pressure becomes more sharply peaked, and, as a result, the effect on the energy release rate is less. At this point, it is unclear what will be the result of considering the pore pressure effect together with the other stabilizing effects of the porous media.

7. CONCLUDING REMARKS

Matched asymptotic expansions have been used to analyse the stress and pore pressure fields near the tip of a shear crack propagating on an impermeable plane in an elastic diffusive (Biot) solid. The analysis complements an earlier one by SIMONS (1977) for a shear crack on a permeable plane. Many of the results are qualitatively similar. One main difference is that a boundary layer is required ahead of the crack-tip on the impermeable plane to accommodate the condition of no flow across the plane ahead of the crack. For the permeable plane, a crack-face boundary layer is needed to meet the condition of no pore pressure change on the crack faces. A second important difference is that the pore pressure at the tip of the crack on the impermeable plane is bounded, but non-zero whereas for the permeable plane the pore pressure approaches zero as the crack-tip is approached through any angle. The results (and those of RUDNICKI and KOUTSIBELAS, 1990) suggest that the pore pressure induced on the crack plane can have a significant effect on the propagation criteria by altering the effective shear resistance, particularly, for more elaborate models of the crack-tip processes. The boundary layer analysis can provide guidance for investigation of these models.

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APPENDIX A: CRACK-TIP INNER EXPANSION

This appendix outlines the solution of the equations describing the crack-tip inner expansion: (30)–(32), subject to (33)–(36). Because the analysis is similar to that in previous work (RICE and SIMONS, 1976; SIMONS, 1977; RUDNICKI and KOUTSIBELAS, 1990), it is presented concisely. Also, to simplify the notation, the superscripts "(1)" and the $\hat{s}$ denoting the crack-tip variables in the text are omitted here.

Let the Fourier transform of a function $f(x, y)$ be defined by

$$F(\kappa, y) = \int_{-\infty}^{\infty} f(x, y) \exp(-\mathrm{i}\kappa x)\,\mathrm{d}x \tag{A1}$$

with inverse

$$f(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} F(\kappa, y) \exp(\mathrm{i}\kappa x)\,\mathrm{d}\kappa, \tag{A2}$$

where $\mathrm{i} = \sqrt{-1}$. The solution to the Fourier transformed field equations (30)–(32) has been given by RICE and SIMONS (1976):

$$\tfrac{1}{2}(\Sigma_{xx} + \Sigma_{yy}) = A(\kappa) \exp[-m(\kappa) y] + B(\kappa) \exp[-n(\kappa) y], \tag{A3}$$

$$\eta P = -\mu A(\kappa) \exp[-m(\kappa) y] - B(\kappa) \exp[-n(\kappa) y], \tag{A4}$$

$$\Sigma_{xy} = -\left[\frac{\mathrm{i}\kappa}{m(\kappa)}C(\kappa)+\mathrm{i}\kappa y A(\kappa)\right]\mathrm{e}^{-m(\kappa)y} - 2n(\kappa)B(\kappa)\,\mathrm{e}^{-n(\kappa)y}, \tag{A5}$$

$$\tfrac{1}{2}(\Sigma_{yy}-\Sigma_{xx}) = [C(\kappa)+A(\kappa)m(\kappa)y]\,\mathrm{e}^{-m(\kappa)y} - \mathrm{i}\{[n^2(\kappa)+\kappa^2]/\kappa\}B(\kappa)\,\mathrm{e}^{-n(\kappa)y}, \tag{A6}$$

where A, B and C are to be determined by the boundary conditions and $m(\kappa)$ and $n(\kappa)$ are defined by the following relations:

$$m^2(\kappa) = \kappa^2, \quad n^2(\kappa) = \kappa^2 - \mathrm{i}\kappa.$$

In order to ensure convergence of the inversion integrals for $y \geqslant 0$, $m(\kappa)$ and $n(\kappa)$ are subject to the following restrictions:

$$\mathrm{Re}\,\{m(\kappa)\} \geqslant 0, \quad \mathrm{Re}\,\{n(\kappa)\} \geqslant 0, \tag{A7}$$

where Re $\{\ldots\}$ stands for "real part of". Application of (A1) to the boundary conditions (33) yields

$$\Sigma_{xy}(\kappa, 0) = 0, \quad \frac{\mathrm{d}P}{\mathrm{d}y}(\kappa, 0) = 0. \tag{A8}$$

However, because the remaining boundary conditions are specified only for $x \leqslant 0$ or $x \geqslant 0$, their Fourier transform are not known. Nevertheless, the Fourier transform of σ_{xy} on $y = 0$ can be written as follows:

$$\Sigma_{xy}(\kappa, 0) = \int_0^{\infty} \sigma_{xy}(x, 0)\exp(-\mathrm{i}\kappa x)\,\mathrm{d}x = F^-(\kappa), \tag{A9}$$

where (35) has been used to eliminate the portion of the integral from $-\infty < x \leqslant 0$ and the last equality defines $F^-(\kappa)$. Because of the behavior of σ_{xy} as $x \to \infty$ given by (36), F^- is an analytic function of κ in the lower half plane, Im $(\kappa) \leqslant 0$. Similarly, the boundary condition (34) can be used to obtain

$$\Sigma_{xx}(\kappa, 0) + 2\eta P(\kappa, 0) = G^+(\kappa), \tag{A10}$$

where

$$G^+(\kappa) = \int_{-\infty}^{0} [\sigma_{xx}(x, 0) + 2\eta p(x, 0)]\exp(-\mathrm{i}\kappa x)\,\mathrm{d}x \tag{A11}$$

is an analytic function of κ in the upper half-plane Im $\{\kappa\} \geqslant 0$. When (A3), (A4), and (A6) are used in (A9) and (A10), the resulting equations can be solved for A, B and C in terms of G^+:

$$A(\kappa) = \frac{G^+(\kappa)}{2(1-\mu)}, \quad B(\kappa) = -\mu\frac{m(\kappa)}{n(\kappa)}\frac{G^+(\kappa)}{2(1-\mu)}, \tag{A12}$$

$$C(\kappa) = \frac{-G^+(\kappa)}{2(1-\mu)}[1 + 2\mu\mathrm{i}\kappa m(\kappa)/n(\kappa)]. \tag{A13}$$

Use of these expressions in (A5) and then substitution into (A9) yields

$$F^-(\kappa) = \frac{G^+(\kappa)}{2(1-\mu)}\left\{\frac{\mathrm{i}\kappa}{m(\kappa)}[1 + 2\mu\mathrm{i}\kappa m(\kappa)/n(\kappa)] + 2\mu m(\kappa)\right\}. \tag{A14}$$

Although neither F^- nor G^+ are known in this equation, the Wiener–Hopf technique (NOBLE, 1988) makes it possible to solve for both quantities by exploiting the analyticity of F^- and G^+ in domains that overlap on Im $(\kappa) = 0$. Execution of the technique requires that the equation (A14) be rewritten such that each side is analytic in a half-plane. To this end, we follow RICE and SIMONS (1976) in decomposing $m(\kappa)$ as the product

$$m(\kappa) = m^+(\kappa)m^-(\kappa), \tag{A15}$$

where $m^+(\kappa) = \kappa^{1/2}$ with its branch-cut on the negative imaginary axis, $-\infty < \mathrm{Im}\,(\kappa) \leqslant 0$, and $m^-(\kappa) = \lim_{\delta\to 0}(\kappa - \mathrm{i}\delta)^{1/2}$ with its branch-cut on the positive imaginary axis, $\delta \leqslant \mathrm{Im}\,(\kappa) < \infty$. For this choice of branch cuts, the first of the restrictions (A7) is met. Thus, as indicated by the superscripts, m^+ is analytic in the upper half-plane, $\mathrm{Im}\,(\kappa) \geqslant 0$, and m^- in the lower half-plane, $\mathrm{Im}\,(\kappa) \leqslant \delta$. Similarly, $n(\kappa)$ can be written as

$$n(\kappa) = m^+(\kappa)n^-(\kappa), \tag{A16}$$

where $n^-(\kappa) = (\kappa - \mathrm{i})^{1/2}$ has its branch-cut on the positive imaginary axis $1 \leqslant \mathrm{Im}\,(\kappa) < \infty$, consistent with the second of (A7), and is analytic for $\mathrm{Im}\,(\kappa) \leqslant 1$.

By using these decompositions, we can rearrange (A14) into the desired form:

$$\frac{m^+(\kappa)G^+(\kappa)}{2(1-\mu)} = \frac{-im^-(\kappa)F^-(\kappa)}{1-2\mu i\kappa[1-m^-(\kappa)/n^-(\kappa)]}, \tag{A17}$$

where we have used the relations $[m^-]^2 = [m^+]^2 = \kappa$. The two sides of (A17) are analytic in regions of the κ plane that overlap on a dense set of points (the real κ axis) and together cover the entire plane. Consequently, each side must be an analytic continuation of the other and represents the same entire function. Because σ_{xy} is expected to be $O(x^{-1/2})$ on $y = 0$ as $x \to 0$ through positive values, $F^- = O(\kappa^{-1/2})$ as $|\kappa| \to \infty$. Consequently, the right-hand side of (A17) is $O(1)$ as $|\kappa| \to \infty$, and so is the left-hand side. By the Liouville theorem (e.g. NOBLE, 1988) the entire function must be a constant, say, k_0 and

$$G^+(\kappa) = 2(1-\mu)k_0/m^+(\kappa). \tag{A18}$$

Substitution into (A12), (A13) and then into (A3)–(A6) yields the solution for the Fourier transformed stress and pore pressure:

$$\tfrac{1}{2}(\Sigma_{xx}+\Sigma_{yy}) = \{k_0/m^+(\kappa)\}\{\mathrm{e}^{-m(\kappa)y} - \mu[m(\kappa)/n(\kappa)]\,\mathrm{e}^{-n(\kappa)y}\}, \tag{A19}$$

$$\eta P = -\mu\{k_0/m^+(\kappa)\}\{\mathrm{e}^{-m(\kappa)y} - [m(\kappa)/n(\kappa)]\,\mathrm{e}^{-n(\kappa)y}\}, \tag{A20}$$

$$\Sigma_{xy} = \frac{k_0}{m^+(\kappa)}\left\{\left[\frac{\mathrm{i}\kappa}{m(\kappa)} - \mathrm{i}\kappa y - 2\mu\frac{\kappa^2}{n(\kappa)}\right]\mathrm{e}^{-m(\kappa)y} + 2\mu m(\kappa)\,\mathrm{e}^{-n(\kappa)y}\right\}, \tag{A21}$$

$$\tfrac{1}{2}(\Sigma_{yy}-\Sigma_{xx}) = \frac{-k_0}{m^+(\kappa)}\left\{[1-m(\kappa)y]\,\mathrm{e}^{-m(\kappa)y} - \mu\frac{m(\kappa)}{n(\kappa)}[(1+2\mathrm{i}\kappa)\,\mathrm{e}^{-n(\kappa)y} - 2\mathrm{i}\kappa\,\mathrm{e}^{-m(\kappa)y}]\right\}. \tag{A22}$$

The task remaining is to invert the Fourier transforms according to (A2). Inversion of terms not involving $n(\kappa)$ can be accomplished using the formula from CARRIER *et al.* (1966) cited by SIMONS (1977). For example, application of this formula yields

$$\int_{-\infty}^{\infty}\left[\frac{1}{m^+(\kappa)} + \frac{1}{m^-(\kappa)}\right]\exp\,[\mathrm{i}\kappa x - m(\kappa)y]\,\mathrm{d}\kappa = 2(\pi/\zeta)^{1/2}\,\mathrm{e}^{+\pi\mathrm{i}/4}, \tag{A23}$$

where $\zeta = r\exp\,(\mathrm{i}\theta)$. As a result the expressions for the mean stress and pore pressure can be written as follows:

$$\tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) = (k_0/2\pi)\{2(\pi/r)^{1/2}\,\mathrm{e}^{-\pi\mathrm{i}/4}\sin\,(\theta/2) - \mu H(x,y)\}, \tag{A24}$$

$$\eta p = -\mu(k_0/2\pi)\{2(\pi/r)^{1/2}\,\mathrm{e}^{-\pi\mathrm{i}/4}\sin\,(\theta/2) - \mu H(x,y)\}, \tag{A25}$$

where H is the inversion integral

$$H(x,y) = \int_{-\infty}^{\infty}[m^-(\kappa)/n(\kappa)]\exp\,[\mathrm{i}\kappa x - n(\kappa)y]\,\mathrm{d}\kappa. \tag{A26}$$

The integral H can be calculated by converting the integration contour in the complex κ plane to one on which the exponent is real and negative, subject to the restrictions (A7). Some details are given in the appendix of RUDNICKI and ROELOFFS (1990) which discusses calculation of a similar integral that arises in the solution for a shear dislocation moving steadily on an impermeable plane in a linear elastic diffusive solid. The result for $H(x, y)$ is given by (41) of the text. The constant k_0 is determined from the order condition (36) to be given by $k_0 = -K^{(e)}(1+i)$. Thus, the final results for the mean stress and pore pressure of the crack-tip solution are given by (37) and (38).

Similarly, (A23) can be used to write the shear stress σ_{xy} and the stress difference $\frac{1}{2}(\sigma_{yy}-\sigma_{xx})$ as (39) and (40) where H is again given by (A26) and (41) and K is the following inversion integral:

$$K(x, y) = \int_{\ell} [1/n \quad (\kappa)] \exp [i\kappa x - m(\kappa) y] \, d\kappa. \tag{A27}$$

This integral cannot be expressed in finite form, but, by evaluating on a contour for which the exponent is real and negative, can be written as (42) of the text.

APPENDIX B: CRACK-LINE SOLUTION

This appendix describes the solution of the equations governing the first term of the crack-line inner expansion. These are (47)–(49) subject to (51)–(53). As in Appendix A, we omit the $\tilde{s}$ and superscript "(1)" in order to simplify the notation.

The first of equations (47) require that σ_{xy} be only a function of x and that function is determined to be $\sigma_{xy}^{(e)}$ by the matching condition as $y \to \infty$. Thus, the solution for σ_{xy} is simply

$$\sigma_{xy} = \sigma_{xy}^{(e)}(x, 0). \tag{B1}$$

Substitution of this result into the second of (47) and integration yields

$$\sigma_{yy} = -y \frac{\partial \sigma_{xy}^{(e)}}{\partial x}(x, 0), \tag{B2}$$

where the function of integration is required to be zero by the second of (50). Integration of (48) yields

$$\sigma_{xx} + \sigma_{yy} + 2\eta p = C_1(x) + C_2(x) y \tag{B3}$$

but $C_1(x) = 0$ from the boundary conditions on $y = 0$. The other function of integration $C_2(x)$ is determined from the matching conditions (52) and (53):

$$C_2(x) = 2(1-\mu) \frac{\partial \sigma^{(e)}}{\partial y}(x, 0), \tag{B4}$$

where $\sigma^{(e)} = (\sigma_{xx}^{(e)} + \sigma_{yy}^{(e)})/2$.

As noted by SIMONS (1977), it is simplest to solve the remaining equation for the quantity $M = [\sigma_{xx} + \sigma_{yy} + (2\eta/\mu) p]$ which is proportional to the change in fluid mass content. Equation (49) indicates that M satisfies the following equation:

$$\frac{\partial^2 M}{\partial y^2} + \frac{\partial M}{\partial x} = 0. \tag{B5}$$

Using (B4) in (B3) and the first of (50) reveals that

$$\frac{\partial M}{\partial y}(x, 0) = C_2(x). \tag{B6}$$

Furthermore, $M = 0$ as $x \to \infty$ and at $x = 0$, the latter because undrained conditions pertain there to order ε. The solution to (B5) subject to these conditions and to (B6) is simply that of the diffusion equation for a prescribed flux at $y = 0$ (Carslaw and Jaeger, 1959, Sec. 2.9):

$$M(x, y) = -\int_x^{\infty} \frac{C_2(x)\exp\left[-y^2/4(\xi - x)\right]}{\left[(\xi - x)\pi\right]^{1/2}}\,\mathrm{d}\xi. \tag{B7}$$

Using this result with (B2), (B3) and (B4) yields the first term of the crack-line expansion given by (56)–(58).