MODELING SLIP ZONES WITH TRIANGULAR DISLOCATION ELEMENTS

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ABSTRACT

We derive in algebraic form the displacement and stress fields produced by a triangular element of constant slip by superposing the solution of Comninou and Dundurs (1975) for an angular dislocation in an elastic half-space. Triangular elements are more flexible for simulating complex geometries than the rectangular elements widely used for modeling slip zones as a superposition of constant slip elements. As an example, we use triangular elements to determine the distribution of slip on a planar surface caused by a prescribed stress drop. Because the slip in elements is uniform, the slip does not taper to zero at the edges of the slipping zone. Consequently, the strain energy in volumes containing the slip zone edge and the stress drop averaged over the slip zone are unbounded. To investigate the effects of these features, we compare our results using uniform slip elements with those from the more elaborate procedure of Wu et al. (1991) that takes proper account of stress singularity at the edge of the slipping zone. The comparison indicates that, for a prescribed uniform stress drop, the uniform slip model slightly overestimates the free surface displacements. The predicted slip surface displacements are more severely overestimated, particularly near the edges of the slipping zone. Nevertheless, extrapolation of the slip surface displacements yields values for the stress intensity factors, the coefficients of the singular stresses near the edge of a crack. The values of stress intensity factors are within 10% of those results obtained by Wu et al. (1991) for the same number of elements.

INTRODUCTION

A prevalent method for investigating subsurface slip is to compare measured surface displacements with those predicted by elasticity solutions for prescribed displacement discontinuities in an elastic half-space. Of these solutions, the most widely used are those for uniform slip in rectangular zones (Chinnery, 1961; Mansinha and Smylie, 1971). The reason for their wide use is that the solutions are obtained in algebraic form, and, consequently, quantities of interest such as stresses or displacements can be obtained without extensive computation. Furthermore, there is no need for integration when rectangular dislocation surfaces are combined to model a region of nonuniform slip; the contributions of the individual rectangles are simply summed. The availability of these solutions for rectangular dislocation surfaces only is, however, a severe limitation. When a slipping zone with a curved boundary is modeled by dividing the region into rectangles, the result is a staircase edge, and curved surfaces that are not cylindrical cannot be simulated easily with rectangles.

In this paper, we construct the solution for a triangular element of uniform discontinuity in a half-space. The solution is obtained by appropriately superposing the solution for an angular dislocation in an elastic half-space derived by Comninou and Dundurs (1975). Because the solution contains that for the rectangular element as a special case, it is inevitably more complicated than
that for a rectangular dislocation surface. Nevertheless, the solution for the triangular element is obtained in algebraic form.

The triangular element is then used in a procedure to determine the distribution of displacement discontinuities (slip) that result when a stress drop is prescribed on a surface interior to the half-space. In brief, the surface is divided into triangles, the traction is evaluated at the center of each triangle, and the resulting linear equations relating stress at these points to the displacement discontinuity are solved. A limitation of this approach is that the assumption of uniform displacement discontinuity in each element precludes the discontinuity from tapering smoothly to zero at the edges of the slipping zone. As a result, the strain energy in volumes containing the edge of the slipping zone is unbounded. Furthermore, because of the strong stress singularity predicted at the edges of an element in which the displacement discontinuity is uniform, the stress drop averaged over the element is unbounded. Consequently, parameters needed to implement simple fracture criteria for the advance of slipping zones, such as stress intensity factors or energy release rates, are difficult to extract from models using such elements. These limitations are common to any approach assuming uniform displacement discontinuity in elements (e.g., Chinnery, 1961).

In order to investigate the effects of these features, we compare the results obtained using uniform displacement discontinuity elements with those obtained from a more elaborate numerical method that takes proper account of the stress behavior at the edge of a slipping zone. Wu et al. (1991) have described a numerical method for modeling slipping zones subjected to prescribed stress drop in an elastic half-space. Because the method incorporates the exact asymptotic form of the displacement discontinuity near the edge of the slipping zone, the difficulties just mentioned for methods using elements with uniform displacement discontinuities are overcome. In particular, accurate results are obtained for the stress intensity factors. The method is, however, computationally more intensive than that using the uniform slip elements. Consequently, part of our motivation for developing the uniform slip triangle was to determine whether satisfactory results can be obtained with it at less computational cost.

The body of the paper begins by describing the construction of the solution for a triangular dislocation element in a half-space from the solution for an angular dislocation in a half-space (Cominou and Dundurs, 1975). The remaining sections describe the use of the element to model zones of prescribed stress drop and comparison of the results with the more elaborate method of Wu et al. (1991).

**Elastic Field of a Triangular Element**

An angular dislocation is shown in Figure 1. The origin of the \( x_i \) coordinate system is on the surface of the half-space with \( x_3 \) directed out of the half-space. The dislocation lies in a vertical plane that makes an angle \( \omega \) with the \( x_2 \) axis and is bounded by the two lines \( PQ \) and \( PR \) that extend to infinity in the negative \( x_3 \) direction. The line \( PQ \) is vertical (parallel to \( x_3 \)) and the line \( PR \) makes an angle \( \alpha \) with the vertical. The coordinates of the intersection point \( P \) are \( \xi \). The positive and negative sides of the dislocation surface are defined by the directions of the arrows on \( QP \) and \( PR \) and the right-hand rule: with the thumb pointing in the direction of the arrows on \( QP \) and \( PR \), the fingers of the right-hand curl, without intersecting the surface of discontinuity, from the negative (−) to positive (+) side of the surface.
The elastic fields of the angular dislocation are the displacements and stresses that result when a uniform displacement discontinuity is prescribed on the dislocation surface:

\[ u^+_i - u^-_i = B_i, \]  

(1)

where \( u^+ \) and \( u^- \) denote the displacements of the positive and negative sides of the surface. The constants \( B_i \) are the components of the Burgers' vector relative to the \( x_i \) coordinate system. The solution to this problem is given by Comninou and Dundurs (1975). They begin with the solution of Yoffe (1960) for an angular dislocation in a full-space. They then use an image dislocation to remove the shear tractions from the surface of the half-space and superposition of the

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**Fig. 1.** Coordinate systems and geometry for an angular dislocation in a half-space. The global coordinate system \( x_i \) has an origin on the surface of the half-space with \( x_3 \) pointing out of the half-space. The angular dislocation \( QPR \) has apex at \( P \), with coordinates \( \xi \) relative to the \( x_i \) system, and lies in a vertical plane making an angle \( \omega \) with the \( x_2 \) axis. The leg \( PQ \) is parallel to \( x_3 \) and the leg \( PR \) makes an angle \( \alpha \) with \( x_3 \). The local coordinate system \( y_i \) has origin at \( P \). The \( y_3 \) axis is parallel to \( x_3 \); \( y_3 \) is perpendicular to the plane of the dislocation and is directed from the negative (−) to the positive (+) side of the surface. The positive and negative sides of the dislocation surface are defined by the directions of the arrows on \( QP \) and \( PR \) and the right-hand rule; with the thumb pointing in the direction of the arrows on \( QP \) and \( PR \), the fingers of the right-hand curl, without intersecting the surface of discontinuity, from the negative (−) to positive (+) side of the surface.
Boussinesq solution (e.g., Sokolnikoff, 1956) to remove the remaining normal tractions from the surface of the half-space. We denote the resulting displacements as follows:

$$u_i = B_j U_{ij}^P (x; \xi, \alpha),$$

(2)

where, except as noted, we employ the summation convention on repeated indices. The expressions for the $U_{ij}^P$ are lengthy and we do not display them here. They can be obtained from the results of Comninou and Dundurs (1975).

Comninou and Dundurs (1975) actually give the solution for the case $\omega = 0$ and express the results in terms of a local coordinate system with origin at the apex of the angular dislocation (the $y_i$ system in Fig. 1). Because it will be simpler for the development that follows, we have chosen to express all the results relative to the global $x_i$ system in Figure 1. The coordinate transformations needed to obtain the $U_{ij}^P$ from the results given by Comninou and Dundurs (1975) are outlined in the Appendix.

The complete stress field is not given by Comninou and Dundurs (1975) but can be obtained from Hook's law:

$$\sigma_{ij} = C_{ijkl} u_{k,l},$$

(3)

where the comma denotes partial differentiation and $C_{ijkl}$ is the tensor of elastic moduli. For the isotropic material considered by Comninou and Dundurs (1975) this tensor is given by

$$C_{ijkl} = \frac{2 \mu}{(1 - 2\nu)} \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

(4)

where $\mu$ is the shear modulus, $\nu$ is Poisson's ratio, and $\delta_{ij}$ is the Kronecker delta.

The next step in the solution for the triangular dislocation element is to superpose two angular dislocations to obtain the solution for a dislocation segment. Figure 2 shows two angular dislocations: QAR, with apex at $\xi^A$, and SBR, with apex at $\xi^B$. As denoted by the counterclockwise arrows, the orientation of SBR is the negative of QAR. Hence, the displacement discontinuities for the two angular dislocations cancel in SBR, and there is a discontinuity only on QABS. The displacement is denoted by

$$u_i = B_j U_{ij}^{AB} (x; \xi^A, \xi^B),$$

(5)

where

$$U_{ij}^{AB} = U_{ij}^A (x; \xi^A, \alpha) - U_{ij}^B (x; \xi^B, \alpha)$$

(6)

and the angles $\alpha$ and $\omega$ that appear in Figure 2 are completely defined by the coordinates of the vertices $\xi^A$ and $\xi^B$. In combining the solutions for the individual angular dislocations, it is necessary to ensure that the multi-valued terms, giving rise to the displacement discontinuity, cancel in the region SBR. It is obvious that the solutions for three differently oriented dislocation segments can be combined to form a triangular dislocation in a vertical plane.
What may not be so obvious is that a similar construction can be used to form a triangular element in an arbitrary oblique plane. The construction of an arbitrary polygonal loop in the infinite elastic body is detailed in Yoffe (1960). Figure 3 shows the superposition of three dislocation segments $DABE$, $EBCF$, and $FCAD$ to form a triangular dislocation element in an oblique plane. The plane of the triangular element lies in plane $x_1x_2$ that dips at an angle $\beta$ from the surface of the half-space. The normal of the plane of triangular element is along the $x_3$ axis. The position and orientation of the triangular element is completely defined by the coordinates of the vertices $\xi^A, \xi^B, \xi^C$ without reference to the angles $\alpha, \beta$, and $\omega$ that appear in Figures 1, 2, and 3. As in combining the angular dislocations to form a dislocation segment, the multivalued terms of the individual dislocation segments must be combined to cancel on the vertical legs and leave a displacement discontinuity only in the triangle $ABC$. 
The solution for the displacements caused by the triangular dislocation element is given by

$$ u_i = B_j U_{ij}(x; \xi^A, \xi^B, \xi^C) $$

(7)

where

$$ U_{ij}(x; \xi^A, \xi^B, \xi^C) = U_{ij}^{AB} + U_{ij}^{BC} + U_{ij}^{CA}. $$

(8)

The stress field can be constructed in similar fashion from the stress field of the angular dislocation solution and will be represented as follows:

$$ \sigma_{ij} = B_k S_{ijk}. $$

(9)

In order to verify our results for the triangular element and their implementation in a computer program, we used two triangular elements to form a rectangle and compared results with solutions for rectangular dislocation ele-
ments. For a vertical dislocation, the solution simplifies sufficiently that it is possible to verify analytically that the displacement fields reduce to those obtained by Chinnery (1961). For dipping surfaces, analytic comparison is not feasible, and we have compared our results for displacements and strains (both on the surface and at depth) with numerical solutions for dipping rectangular slip surfaces furnished by Simpson (personal comm., 1990). Simpson’s results were obtained using a program described by Erickson (1986) based on equations derived by Converse (unpublished manuscript, 1973). Our results for the maximum free surface displacement and strain differ by less than 0.1% from those furnished by Simpson. Similar agreement is achieved at interior points, except near singularities.

Solutions for nonuniform distributions of displacement discontinuities can be obtained by superposition of (7) and (9). If a region of nonuniform displacement discontinuity is approximated by \( N \) triangular elements of uniform displacement discontinuity, the resulting displacement and stress are given by

\[
\begin{align*}
    u_i(x) &= \sum_{L=1}^{N} U_{ij}(x, \xi^L) B^L_j, \\
    \sigma_{ij}(x) &= \sum_{L=1}^{N} S_{ijk}(x, \xi^L) B^L_k,
\end{align*}
\]

where \( B^L_j \) is the displacement discontinuity in the \( L \)th triangle, \( \xi^L \) denotes the coordinates of the vertices of the triangle and \( U_{ij} \) and \( S_{ijk} \) are the functions that appear in (7) and (9).

**MODELING SLIP ZONES OF PRESCRIBED STRESS DROP**

If a surface of displacement discontinuity can be approximated by a mosaic of triangular elements of constant displacement discontinuity, then (10) and (11) provide solutions for the resulting displacement and stress fields in the half space. Alternatively, the distribution of displacement discontinuity corresponding to a given traction change (stress drop) on the surface can be calculated approximately. In this section, we describe the solution to this second problem. In the following section we summarize a more elaborate procedure for determining the elastic fields due to zones of prescribed stress drop (Wu et al., 1991). Then we compare the results obtained by the two methods.

We assume that the slipping zone is approximated by \( N \) triangular elements, and we choose the element centroids as collocation points. The stress at these points is given by

\[
\sigma^M_{ij} = \sum_{L=1}^{N} S^M_{ijk} B^L_k,
\]

where \( \sigma^M_{ij} = \sigma_i(x^M), S^M_{ijk} = S_{ijk}(x^M, \xi^L) \) and \( x^M \) denotes the coordinates of the center of the \( M \)th element. Hence, the traction vector at the center of the \( M \)th element is

\[
t^M_i = \sum_{L=1}^{N} a^M_{ik} B^L_k,
\]
where \( \alpha_{ik}^{M} = S_{i,k}^{M} n_{j}^{M} \) (no summation on \( M \)) and \( n_{j}^{M} \) are the components of the normal vector of the \( M \)th element. The repeated application of (13) for all \( N \) centroids produces the following system of linear algebraic equations:

\[
[t] = [A][B],
\]

(14)

where \([t]\) is a column vector containing the dislocation stress vector at the center of the elements, \([B]\) is a column vector containing the displacement discontinuity on the surface element, and \([A]\) is the matrix containing the geometric coefficients. For a prescribed traction, the displacement discontinuity is obtained by solving the system of linear equations. Once the displacement discontinuity is known for each element, the elastic field at any point may be evaluated.

Because the displacement discontinuity in the triangular elements is uniform, the stress near the edge of the element is singular. More specifically, the stress near the edge of the element is expected to approach the \( \rho^{-1} \) behavior (where \( \rho \) is the distance from the edge) exhibited by straight dislocation lines. Because of this strong singularity, the stress drop averaged over the dislocation element is unbounded and the strain energy change caused by the introduction of a triangular element of uniform displacement discontinuity is unbounded. Indeed, because of these features, there is some question whether the procedure summarized by (13) will converge as the element size becomes smaller. Nevertheless, experience with triangular dislocation elements in a full space indicates that no problems arise if the stress collocation points are taken at the centroids of the triangles (Barnett, personal comm., 1991; Wong, 1985).

In contrast to the behavior of the stress near the edge of a dislocation element, the stress near the edge of a crack in a linear elastic body is well known to be of the order of \( \rho^{-1/2} \) (e.g., Lawn and Wilshaw, 1975). For a crack, the stress drop, which is prescribed, is of course bounded and so is the strain energy change. The differing behavior of the stresses near the edge of dislocation elements and cracks suggests that the procedure using the triangular dislocation elements will not accurately model the stresses near the edge of a slipping zone. Nevertheless, the procedure may be sufficiently accurate for determining the average stress drop and free surface displacements. In order to evaluate this conjecture, we compare results using the triangular dislocation element with a more elaborate procedure of Wu et al. (1991) for determining the displacement due to surfaces subjected to prescribed stress drop in an elastic half-space. Before doing this, we first outline briefly the procedure of Wu et al. (1991).

**More Elaborate Procedure for Stress Drop Modeling**

The analysis of Wu et al. (1991) begins with the integral representation of the displacements in an elastic body caused by a specified displacement discontinuity \( B_i(\xi) \) on a surface \( S \) (e.g., Steketee, 1958):

\[
u_i(x) = \int_{S} C_{ihmn} n_m B_n(\xi) \frac{\partial G_{ih}(x, \xi)}{\partial \xi_l} d\xi.
\]

(15)

In this expression, \( n \) is the unit normal directed from the \((-)\) side to the \((+)\) side of \( S \), and \( G_{ih}(x, \xi) \) is the displacement in the \( i \)th direction at \( x \) due to a unit
point force in the \( j \)th direction at \( \xi \). Substitution of (15) into (3) yields an expression for the stresses. If the tractions are required to have a prescribed value (stress drop) \( p_j \) on \( S \), then the stresses must satisfy the condition

\[
n_i \sigma_{ij} = -p_j
\]  

(16)

for \( x \) on \( S \). Substitution of the stresses in (16) yields the following integral equation for the unknown displacement discontinuity distribution, \( B_n(\xi) \):

\[
\int_S K_{mn}(x, \xi) B_n(\xi) \, d\xi = -p_m(x)
\]  

(17)

for \( x \) on \( S \), where

\[
K_{mn} = n_i C_{ijkl} C_{pqmn} \frac{\partial^2 G_{kp}}{\partial x_i \partial q_j}
\]  

(18)

Equations (13) or (14) can be considered discrete versions of (17) obtained by dividing the surface \( S \) into triangular elements, approximating \( B_n(\xi) \) as constant in each triangle, and evaluating the stress at the centroids of the triangles. As already noted, this procedure results in a stress singularity at the edges of \( S \) that is stronger than that at the edge of a crack. In contrast, Wu et al. (1991), adopting the method of Lee and Keer (1986), approximate the \( B_n(\xi) \) in a way that builds in the proper stress singularity at the edge of the slipping zone. More specifically, Wu et al. (1991) assume that the \( B_n(\xi) \) have the following form:

\[
B_n(\xi) = [2a \varepsilon - \varepsilon^2]^{1/2} f_n(\xi),
\]  

(19)

where \( \varepsilon \) is the shortest distance of the integration point \( \xi \) to the crack front, \( a \) is the largest dimension of crack surface \( S \) (Murakami and Nemat-Nasser, 1983), and the \( f_n(\xi) \) are taken as constant in each triangle. Assuming the displacement discontinuities in the form (19) makes it possible to determine the stress intensity factors accurately. These factors are defined as follows (e.g., Broek, 1986):

\[
K_L = \lim_{\rho \to 0} (2\pi \rho^{1/2} n_i \sigma_{iL}),
\]  

(20)

where \( \rho \) is the distance from the crack edge, \( L = I \) corresponds to the opening (tensile) mode, \( L = II \) corresponds to shearing perpendicular to the crack edge (plane strain), and \( L = III \) corresponds to shearing parallel to the crack edge (antiplane strain). In practice, the stress intensities factors are not determined from (20) but from the relative displacements near the edge of the crack. These relative displacements near the edge of the crack have the following asymptotic form (e.g., Rice, 1968):

\[
(B_I, B_{II}, B_{III}) = \frac{4(1-v)}{\mu} \left( \frac{\rho}{2\pi} \right)^{1/2} (K_I, K_{II}, K_{III}/(1-v))
\]  

(21)
as \( \rho \to 0 \). The stress intensity factors \( K_\ell \) can be determined from (19) by expressing this equation relative to coordinates parallel and perpendicular to the crack edge and comparing with (21).

**Comparison of Results for Stress Drop Models**

In order to evaluate the effectiveness of using the uniform displacement discontinuity triangles to model slipping zones of prescribed stress drop, we compare results obtained from (13) with those obtained using the method of Wu et al. (1991). Although both methods can be applied to arbitrarily shaped slipping zones and to arbitrary distributions of stress drop, the comparison is based on results for elliptical slipping zones with uniform stress drop.

The geometry of the slip zone is shown in Figure 4. The dip angle from the free surface is \( \beta \), the depth of the center of slip zone is \( h \), the depth of the shallowest portion of the slip zone is \( d \), and \( a \) and \( b \) are the major and minor semi-axes of the ellipse. The stress drop is assumed to be a uniform shear of magnitude \( \Delta \tau \) in the normal dip-slip sense; the stress change normal to the slip surface is prescribed as zero. The origin of the \( x \) coordinate system is located on the free surface directly over the center of the slip zone with \( x_3 \) pointing out of the free surface. It will be convenient to refer also to the \( z \) coordinate system with origin at the center of the slip zone. The \( z_1 \) direction is parallel to \( x_1, x_2 \) points down-dip, and \( z_3 \) is perpendicular to the slip surface and points from the \((-)\) to the \((+)\) side of the slip surface. Polar coordinates in the plane of the slip surface are denoted by \((r, \phi)\). Poisson’s ratio is taken as 0.25 in all the calculations.

Figure 5 compares the free surface displacements along \( x_1 = 0 \) predicted by the uniform slip element procedure and the method of Wu et al. (1991). Figure 5a shows the vertical displacements and Figure 5b shows the horizontal. Results using 152 elements are for a circular slip zone \((a = b)\), depth \( h = 2a\), and three values of dip angle: 15°, 45°, and 75°. As shown, the results obtained from the uniform slip triangles slightly overestimate the free surface displacement. This is to be expected since the slip surface displacements are not required to taper to zero at the edges of the slipping zone, as they are in the method of Wu et al. (1991). In other words, for a given stress drop, the uniform slip triangles predict a slightly larger moment. The maximum displacements are, however, less than 8% greater than predicted by the method of Wu et al. (1991).

Figure 6 compares the slip surface displacements predicted using the uniform slip element with those predicted by the more elaborate approach of Wu et al. (1991) for a shallow \((d/a = 0.1)\), elliptical \((a/b = 2)\) slip zone dipping at 45°. The downdip displacement is shown along the principal axes of the ellipse, \( z_1 = 0 \) (Fig. 6a) and \( z_2 = 0 \) (Fig. 6b). Results for the uniform slip elements are shown for a coarse mesh (52 elements) and a fine mesh (200 elements); the results for the approach of Wu et al. (1991) are based on 168 elements.

Application of a uniform stress drop to an elliptical slip zone in an infinite elastic medium would cause an exactly elliptical distribution of relative displacement (Kostrov and Das, 1984). The results of Wu et al. (1991) show a roughly elliptical distribution with some distortion due to the proximity of the free surface. The results obtained with the uniform slip elements for the finer mesh are similar to those of Wu et al. (1991). This agreement is better toward the center of the slip zone than at the edges. In particular, the results using the
uniform slip triangles do not approach zero at the edges of the slipping zone. The results for the uniform slip triangles consistently overestimate the magnitude of the slip surface displacements. This appears to occur because of the strong variation of stress with the uniform slip triangle: The stress is strongly singular at the edge of the element but decreases rapidly toward the center. Since the collocation point is taken at the centroid of the triangle, where the stress is lowest, the coefficients $\alpha^{MM}_{ij}$ in (13) are underestimated relative to the contributions from other elements. This underestimation causes an overprediction of the slip surface displacements for a given stress drop.
Fig. 5. Comparison of the free surface displacements predicted using the constant slip elements and the method of Wu et al. (1991). Results are shown for an inclined circular slip zone ($\alpha = b, \frac{h}{\alpha} = 2$) subjected to uniform stress drop $\Delta \tau$ in the normal dip-slip sense. The number of triangular elements used was 152 in each case. The vertical ($u_3$) and horizontal displacements ($u_2$), divided by $\alpha\Delta \tau / \mu$, on the line $x_1 = 0$ are shown for three values of the dip angle $\beta$: 15°, 45°, and 75°.
Fig. 6. The relative displacements on the slip surface $R_2$, divided by $a \Delta \tau/\mu$, for a shallow elliptical slip zone ($a/b = 2, d/a = 0.1, \beta = 45^\circ$) subjected to uniform stress drop $\Delta \tau$ in the reverse dip-slip (thrust) sense. The results using the uniform slip elements are shown for a coarse mesh of 52 elements and a fine mesh of 200 elements and are compared with results using the method of Wu et al. using 168 elements. The relative displacement on the slip surface (slip) in the $x$ direction is...
The method of Wu et al. (1991) is designed so that the stress and displacement fields at the edges of the slipping zone are the appropriate ones for a crack in an elastic body. Consequently, the stress intensity factors can be estimated more accurately. Because the relative displacements determined by the uniform slip triangles do not have the form (21) and, in particular, do not approach zero at the crack edge, the stress intensity factors cannot be determined from (21) in the limit as \( \rho \) approaches zero. Nevertheless, it has been suggested (e.g., Chan et al., 1970; Wong, 1985) that the stress intensity factors can be determined by extrapolation of the relative displacements. The accuracy of this procedure is evaluated by applying it to the results obtained in Figure 6 with 200 elements.

Figure 7 plots the absolute values of relative displacements in the \( z_2 \) direction, \( B_2 \), in nondimensional form along the principal semi-axes. If \( B_2 \) had the form specified by (21), this ratio would equal \( K_{II}/\Delta \tau (\pi a)^{1/2} \) on the minor principal axis \( (\phi = \pm 90^\circ; z_2 = 0) \) and \( K_{II}(1 - \nu)/\Delta \tau (\pi a)^{1/2} \) on the major principal axis \( (\phi = 0; z_2 = 0) \). Results along the major principal axis are plotted against \( \rho/a \), and those on the minor principal axis are plotted against \( \rho/b \). The nondimensional combination plotted will be proportional to the stress intensity factors if the slip surface displacements vary as \( \rho^{-1/2} \), as \( \rho \) approaches zero. Extrapolation of the linear portion of the plot to \( \rho = 0 \), as shown by the dashed

![Graph](image)

**Fig. 7.** The absolute values of the relative displacements shown in Figure 6 (for 200 elements) are plotted in nondimensional form against the distance away from the edge of the slip surface \( \rho \). The nondimensional combination plotted will be proportional to the stress intensity factors if the slip surface displacements vary as \( \rho^{-1/2} \), as \( \rho \) approaches zero. Values of \( \phi \) refer to the polar angle shown in Figure 4. Extrapolation of the linear portion of the plot to \( \rho = 0 \), as shown by the dashed lines yields estimates of the stress intensity factors. The values are 1.70 (\( \phi = -90^\circ \)), 1.23 (\( \phi = 0^\circ \)), and 1.66 (\( \phi = 90^\circ \)).
lines, yields estimates of the stress intensity factors. The values are 1.70 (\(\phi = -90^\circ\)), 1.23 (\(\phi = 0^\circ\)), and 1.66 (\(\phi = 90^\circ\)). The corresponding values from the more elaborate method of Wu et al. (1991) are 1.64, 1.18, and 1.61, respectively. The difference between the values for \(\phi = \pm 90^\circ\) is due to the presence of the free surface; the values would be identical in an infinite body. Although the results obtained using the uniform dislocation triangles consistently overestimate the values of stress intensity factors, they are within 5% of those results obtained by Wu et al. (1991).

CONCLUSIONS

We have used the solution of Comninou and Dundurs (1975) for an angular dislocation in a half-space to construct the stress and displacement fields for a triangular dislocation element in a half-space. Although the expressions are lengthy, and hence not displayed here, they are in algebraic form. Consequently, no integration is needed when a surface of relative displacement, either slip or opening, is approximated by dividing the surface into triangular elements in which the relative displacement is constant: the contributions from the individual triangles can simply be summed. The triangular element is considerably more versatile than the widely used rectangular elements in approximating slipping zones that have curved boundaries or are nonplanar.

We have also discussed and implemented a procedure for using triangular dislocation elements to determine the relative displacements on a surface for which the stress drop is prescribed. The relative displacements are determined by inversion of a matrix that is formed by summing the contributions from individual triangular elements. Because the relative displacement in the elements is assumed to be constant, the procedure is incapable of predicting relative displacements that taper to zero at the edges of the zone of stress drop. Consequently, we have compared the results with a more elaborate procedure of Wu et al. (1991) that builds in the proper stress and relative displacement fields at the edges of a crack in an elastic solid. For a given uniform stress drop, the two approaches predict displacements of the free surface that differ very little, although the method using the uniform relative displacements triangles does overestimate the maximum displacement. There is a greater discrepancy in the prediction of the relative displacements on the slip surface, particularly near the edges of the slipping zone. The discrepancy is, however, reduced by using a larger number of elements. Furthermore, the stress intensity factors predicted by extrapolation of the relative displacements appear to be less than about 10% higher than those obtained by the method of Wu et al. (1991) with a similar number of elements.

Because the method using uniform relative displacement triangles is sufficiently accurate and much more economical than the method of Wu et al. (1991), it appears to be better suited for inferring the geometry and distribution of stress drop from observations of surface displacements. Once they have been determined, the method of Wu et al. (1991) can be applied to obtain accurate estimates of the stress intensity factors, and these can be used to evaluate conditions leading to the cessation of rupture and the propensity for further propagation.

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REFERENCES


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APPENDIX

In the body of the paper, the results for the displacements and stresses due to the angular dislocation in a half-space are expressed relative to the global coordinate system $x_i$ with origin on the half-space surface (Fig. 1). Although this simplifies the construction of the solution for the triangular element, Comninou and Dundurs (1975) actually give the solution for the angular dislocation in terms of the local coordinate system $y_i$ in Figure 1. This appendix outlines the coordinate transformations that are needed to obtain the solution in the form given in the body of the paper from that given by Comninou and Dundurs (1975).

The origin of the $y_i$ system shown in Figure 1 is at $P$, the apex of the angular dislocation, with coordinates $\xi_i$ relative to the $x_i$ system. The $y_3$ axis is parallel
to the $x_3$ axis; the $y_1$ and $y_2$ axes are rotated from the $x_1$ and $x_2$ axes by an angle $\omega$ measured clockwise from the positive $x_3$ axis. Thus, the $y_i$ system is related to the $x_i$ system as follows:

$$y_i = \Omega_{ij}(x_j - \xi_j),$$  \hspace{1cm} (A1)

where

$$\Omega_{ij} = \begin{bmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (A2)

is the matrix of components of the rotation tensor.

As shown in Figure 1, the angular dislocation lies in the $y_2 y_3$ plane and the vertical leg $QP$ coincides with the negative $y_3$ axis. The solution for the displacement components $v_i$ relative to the $y_i$ system, as given by Comninou and Dundurs (1975), can be represented as follows:

$$v_i = b_i V_{ij}(y;|\xi_3|, \alpha),$$  \hspace{1cm} (A3)

where the $b_i$ are the components of the displacement discontinuity relative to the $y_i$ system. The displacement functions $V_{ij}(y;|\xi_3|, \alpha)$ depend on the dip angle of the angular dislocation ($\alpha$) and the depth of the corner $P$ below the free surface ($|\xi_3|$).

The displacement components $u_i$ relative to the global $x_i$ system are related to the $v_i$ by

$$u_n = \Omega_{mn} v_n.$$  \hspace{1cm} (A4)

Similarly, the components of the displacement discontinuity $B_i$ relative to the $x_i$ system are related to the $b_i$ by

$$b_k = \Omega_{ik} B_i.$$  \hspace{1cm} (A5)

Substituting (A4) and (A5) into (A3) and comparing with (2) yields

$$U_{ij}^\alpha(x; \xi, \alpha) = \Omega_{ik} \Omega_{jm} V_{km}(y;|\xi_3|, \alpha).$$  \hspace{1cm} (A6)

When two angular dislocations are combined to form a dislocation segment (see Equations 5 and 6), the rotation matrix $\Omega_{ij}$ will be the same for each dislocation. The rotation matrix will, however, be different for the different dislocation segments making the triangular element. The transformation for the stress field can be constructed in a similar fashion.