Localization in Undrained Deformation

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Background (Rice, JGR, 1975)

\[ \frac{d\gamma}{G} = \frac{1}{H} [d\tau - \mu d(\sigma - p)] \]

\[ d\varepsilon = -\frac{1}{M} d(\sigma - p) + \frac{\beta}{H} [d\tau - \mu d(\sigma - p)] \]

\[ d^\rho \varepsilon = \beta d^\rho \gamma \]

Equilibrium: \( \frac{\partial \tau}{\partial y} = 0 \)

Fluid mass conservation: \( \frac{\partial}{\partial y} \left( \frac{q}{\rho} \right) + \frac{\partial \varepsilon}{\partial t} = 0 \)

and Darcy's Law: \( q = -\rho \kappa \frac{\partial p}{\partial y} \Rightarrow \)

\[ \frac{\partial}{\partial y} \left[ \kappa \frac{\partial p}{\partial y} \right] = \frac{\partial \varepsilon}{\partial t} \]
Geometric Interpretation of Constitutive Parameters

\[ d\varepsilon^p = \beta d\bar{\gamma}^p > 0 \quad \text{(dilation)} \]

\[ -d\varepsilon^p \quad \text{(compaction)} \]

\[ \sigma = \text{constant} \]

\[ H_{\tan} = H / (1 + H / G) \]
undrained ($\beta\mu > 0$) response curve

inc $\sigma - p = \text{const}$, $\mu > 0$, $\beta > 0$

dec $\sigma - p = \text{const}$, $\mu < 0$, $\beta < 0$

drained ($\sigma - p = \text{const}$) response curve
Localization predicted to occur here for *undrained* response.

**BUT**

Rice showed that small spatial perturbations grow exponentially in time beginning here.

Localization predicted to occur here for *drained* response.
$$\beta \mu < 0$$

(compaction on frictional y.s. or dilation on cap y. s.)

In the diagram:
- Drained response curve
- Undrained response curve
- **Inc** $\sigma - p = \text{const, } \mu < 0, \beta > 0$
- **Dec** $\sigma - p = \text{const, } \mu > 0, \beta < 0$
Layer Problem is special because

- Localization predicted to occur at peak of both drained and undrained response curves
- Band formation is parallel to the layer boundaries
- More generally, there is a strong dependence on deviatoric stress state (Lode angle).
Generalize:

Constitutive Relation (drained, \( p = \text{constant} \))

\[
d\varepsilon_{ij}^{el} = \left\{ \frac{d\sigma_{ij} - \nu}{(1+\nu)}\delta_{ij} d\sigma_{kk} \right\} / 2G
\]

\[
d\varepsilon_{ij}^{in} = \frac{1}{h} \left( \frac{s_{ij}}{2\tau} + \frac{1}{3} \beta \delta_{ij} \right) \left( \frac{s_{kl}}{2\tau} + \frac{1}{3} \mu \delta_{kl} \right) d\sigma_{kl}
\]

\[
\bar{\tau} = (s_{ij}s_{ij} / 2)^{1/2}, \quad s_{ij} = \sigma_{ij} - (\sigma_{kk} / 3) \delta_{ij}
\]

\( \mu \) is a friction coefficient = \( d\tau_{\text{yield}} / d\sigma \), \( \sigma = -(\sigma_{kk} / 3) \)

\( \beta \) is a dilatancy factor = \( d\varepsilon_{kk}^{in} / d\bar{\gamma}^{in} \), \( d\bar{\gamma}^{in} = (2de_{ij}^{in}de_{ij}^{in})^{1/2} \)

\[
de_{ij}^{in} = d\varepsilon_{ij}^{in} - (d\varepsilon_{kk}^{in} / 3) \delta_{ij}
\]

\[
h = d\bar{\tau} / d\bar{\gamma}^{in}, \quad \sigma = \text{constant}
\]

\[
h_{\text{tan}} = d\bar{\tau} / d\bar{\gamma} = h / (1 + h / G)
\]
Localization condition yields $h_{\text{crit}}$ and $\theta_{\text{band}}$

$$\tau = \frac{h_{\text{crit}}}{1 + h_{\text{crit}} / G}$$

$\sigma = \text{constant}$

$\hbar_{\text{crit}}$ and $\theta_{\text{band}}$

axisymmetric compression

axisymmetric extension

$2\sin\theta_{\text{Lode}}$

$2\sin\theta_{\text{Lode}}$

$\beta = 0.3, \mu = 0.6$

$\beta = -0.3, \mu = 0.6$

$\beta = 0.3, \mu = -0.6$

$\beta = -0.3, \mu = -0.6$
Questions

• How does band angle predicted for undrained deformation differ from prediction for drained?
• How it depend on deviatoric stress state (Lode angle)?
• How do critical hardening moduli for drained and undrained deformation depend on stress state?

Will also turn out that predictions for undrained deformation depend strongly on poroelastic parameters.
Undrained Response

...has the same form as drained if (Rudnicki, 1985)

(i) the elastic portion of strain increments can be described by linear, isotropic poroelasticity;

(ii) the role of the pore pressure in the inelastic strain increments is included by replacing the stress by the Terzaghi form of the effective stress,
\[ \sigma \rightarrow \sigma - p \]

(iii) the inelastic increment in the apparent void volume fraction is equal to the inelastic volume strain increment.
\[ dV^{in} = d\varepsilon^{in} \]

Rice [1977] has argued on theoretical grounds that (ii) and (iii) are appropriate for geomaterials in which the primary mechanisms of inelastic deformation are microcracking from sharp-tipped flaws and frictional sliding on surfaces with small real areas of contact. In addition, (ii) seems to be supported by experiments.
To get the undrained response from the drained, make the following substitutions in expression for inelastic strain increments:

\[(\mu, \beta) \rightarrow (1 - B)(\mu, \beta)\]
\[h \rightarrow H = h + (\mu \beta KB / \zeta)\]
where \(B = \) Skempton's coefficient, \(\zeta = 1 - K / K'_s\), Biot's coefficient (often \(\alpha\))
where \(K\) is drained elastic bulk modulus for porous solid
\(K'_s\) is related to bulk modulus of solid constituents

For elastic strain increments replace Poisson’s ratio by its undrained value:

\[\nu_u = \frac{3\nu + (1 - 2\nu)B\zeta}{3 - (1 - 2\nu)B\zeta}, \quad \left(\frac{1}{2}, \text{ for } B = 1, \zeta = 1\right)\]

Note: for \(B = 1\) (incompressible solid and fluid constituents, soil mechanics approx), effective value of \(\mu\) and \(\beta\) vanish; i.e., inelastic pressure dependence and volume change vanish (Runesson et al., 1998)
Dependence on Lode angle and poroelastic parameters for $\mu = 0.6$, $\beta = 0.3$

$N = 2 \sin(\theta_{Lode})$

$h_{\text{crit}}/G = \text{critical hardening modulus for } \text{drained deformation}$

$H_{\text{crit}}/G = \text{critical hardening modulus for undrained deformation}$

$h_{\text{crit\ und}}/G = \text{critical hardening modulus of underlying drained response curve at undrained response satisfies localization condition.}$

Note: localization condition is met for undrained response before drained response.
Dependence on Lode angle and poroelastic parameters for $\mu = 0.6$, $\beta = -0.3$

$h_{\text{crit}}/G = $ critical hardening modulus for \textit{drained} deformation

$H_{\text{crit}}/G = $ critical hardening modulus for undrained deformation

$h_{\text{crit}}^{\text{und}}/G = $ critical hardening modulus of underlying drained response curve at undrained response satisfies localization condition.
Conclusions and Caveats

- Relation between localization conditions and band angle for drained and undrained deformation depend strongly on deviatoric stress state (Lode angle) and poroelastic parameters.
- Band angle decreases as solid and fluid constituents become less compressible.
- For a Drucker Prager (Rudnicki Rice) constitutive relation undrained response (and hence localization conditions) can be obtained directly by substitution (no need to do separate analysis).
- Relevance of undrained localization condition is unclear when it occurs after the drained condition.
- Analysis should really include possibility of spatial perturbations (as in Rice, 1975) for which local and global drainage conditions will differ.